International Competition in Mathematics for Universtiy Students in Plovdiv, Bulgaria 1996

PROBLEMS AND SOLUTIONS

First day — August 2, 1996

Problem 1. (10 points)

Let for $j = 0, \dots, n$, $a_j = a_0 + jd$, where a_0, d are fixed real numbers. Put

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ a_1 & a_0 & a_1 & \dots & a_{n-1} \\ a_2 & a_1 & a_0 & \dots & a_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ a_n & a_{n-1} & a_{n-2} & \dots & a_0 \end{pmatrix}.$$

Calculate $\det(A)$, where $\det(A)$ denotes the determinant of A.

Solution. Adding the first column of A to the last column we get that

$$\det(A) = (a_0 + a_n) \det \begin{pmatrix} a_0 & a_1 & a_2 & \dots & 1 \\ a_1 & a_0 & a_1 & \dots & 1 \\ a_2 & a_1 & a_0 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ a_n & a_{n-1} & a_{n-2} & \dots & 1 \end{pmatrix}.$$

Subtracting the n-th row of the above matrix from the $(n+1)$-st one, $(n-1)$-st from n-th, $\dots$, first from second we obtain that

$$\det(A) = (a_0 + a_n) \det \begin{pmatrix} a_0 & a_1 & a_2 & \dots & 1 \\ d & -d & -d & \dots & 0 \\ d & d & -d & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ d & d & d & \dots & 0 \end{pmatrix}.$$

Hence,

$$\det(A) = (-1)^n (a_0 + a_n) \det \begin{pmatrix} d & -d & -d & \dots & -d \\ d & d & -d & \dots & -d \\ d & d & d & \dots & -d \\ \dots & \dots & \dots & \dots & \dots \\ d & d & d & \dots & d \end{pmatrix}.$$

Adding the last row of the above matrix to the other rows we have

$$\det(A) = (-1)^n(a_0+a_n)\det\begin{pmatrix} 2d & 0 & 0 & \dots & 0 \\ 2d & 2d & 0 & \dots & 0 \\ 2d & 2d & 2d & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ d & d & d & \dots & d \end{pmatrix} = (-1)^n(a_0+a_n)2^{n-1}d^n.$$

Problem 2. (10 points)
Evaluate the definite integral

$$\int_{-\pi}^{\pi} \frac{\sin nx}{(1+2^x)\sin x}dx,$$

where n is a natural number.

Solution. We have

$$\begin{aligned} I_n &= \int_{-\pi}^{\pi} \frac{\sin nx}{(1+2^x)\sin x}dx \\ &= \int_{0}^{\pi} \frac{\sin nx}{(1+2^x)\sin x}dx + \int_{-\pi}^{0} \frac{\sin nx}{(1+2^x)\sin x}dx. \end{aligned}$$

In the second integral we make the change of variable $x = -x$ and obtain

$$\begin{aligned} I_n &= \int_{0}^{\pi} \frac{\sin nx}{(1+2^x)\sin x}dx + \int_{0}^{\pi} \frac{\sin nx}{(1+2^{-x})\sin x}dx \\ &= \int_{0}^{\pi} \frac{(1+2^x)\sin nx}{(1+2^x)\sin x}dx \\ &= \int_{0}^{\pi} \frac{\sin nx}{\sin x}dx. \end{aligned}$$

For $n \geq 2$ we have

$$\begin{aligned} I_n - I_{n-2} &= \int_{0}^{\pi} \frac{\sin nx - \sin (n-2)x}{\sin x}dx \\ &= 2\int_{0}^{\pi} \cos (n-1)x dx = 0. \end{aligned}$$

The answer

$$I_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \pi & \text{if } n \text{ is odd} \end{cases}$$

follows from the above formula and $I_0 = 0$, $I_1 = \pi$.

Problem 3. (15 points)

The linear operator A on the vector space V is called an involution if $A^2 = E$ where E is the identity operator on V. Let $\dim V = n < \infty$.

(i) Prove that for every involution A on V there exists a basis of V consisting of eigenvectors of A.

(ii) Find the maximal number of distinct pairwise commuting involutions on V.

Solution.

(i) Let $B = \frac{1}{2}(A+E)$. Then

$$B^2 = \frac{1}{4}(A^2 + 2AE + E) = \frac{1}{4}(2AE + 2E) = \frac{1}{2}(A+E) = B.$$

Hence B is a projection. Thus there exists a basis of eigenvectors for B, and the matrix of B in this basis is of the form $\mathrm{diag}(1, \dots, 1, 0, \dots, 0)$.

Since $A = 2B - E$ the eigenvalues of A are ± 1 only.

(ii) Let $\{A_i : i \in I\}$ be a set of commuting diagonalizable operators on V, and let A_1 be one of these operators. Choose an eigenvalue λ of A_1 and denote $V_\lambda = \{v \in V : A_1 v = \lambda v\}$. Then V_λ is a subspace of V, and since $A_1 A_i = A_i A_1$ for each $i \in I$ we obtain that V_λ is invariant under each A_i. If $V_\lambda = V$ then A_1 is either E or $-E$, and we can start with another operator A_i. If $V_\lambda \neq V$ we proceed by induction on $\dim V$ in order to find a common eigenvector for all A_i. Therefore $\{A_i : i \in I\}$ are simultaneously diagonalizable.

If they are involutions then $|I| \leq 2^n$ since the diagonal entries may equal 1 or -1 only.

Problem 4. (15 points)

Let $a_1 = 1$, $a_n = \frac{1}{n}\sum_{k=1}^{n-1} a_k a_{n-k}$ for $n \geq 2$. Show that

(i) $\limsup_{n\to\infty} |a_n|^{1/n} < 2^{-1/2}$;

(ii) $\limsup_{n\to\infty} |a_n|^{1/n} \geq 2/3$.

Solution.

(i) We show by induction that

$$(*) \qquad a_n \leq q^n \quad \text{for} \quad n \geq 3,$$

where $q = 0.7$ and use that $0.7 < 2^{-1/2}$. One has $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$, $a_4 = \frac{11}{48}$. Therefore (∗) is true for $n = 3$ and $n = 4$. Assume (∗) is true for $n \le N-1$ for some $N \ge 5$. Then

$$a_N = \frac{2}{N}a_{N-1}+\frac{1}{N}a_{N-2}+\frac{1}{N}\sum_{k=3}^{N-3} a_k a_{N-k} \le \frac{2}{N}q^{N-1}+\frac{1}{N}q^{N-2}+\frac{N-5}{N}q^N \le q^N$$

because $\frac{2}{q} + \frac{1}{q^2} \le 5$.

(ii) We show by induction that

$$a_n \ge q^n \quad \text{for} \quad n \ge 2,$$

where $q = \frac{2}{3}$. One has $a_2 = \frac{1}{2} > \left(\frac{2}{3}\right)^2 = q^2$. Going by induction we have for $N \ge 3$

$$a_N = \frac{2}{N}a_{N-1} + \frac{1}{N}\sum_{k=2}^{N-2} a_k a_{N-k} \ge \frac{2}{N}q^{N-1} + \frac{N-3}{N}q^N = q^N$$

because $\frac{2}{q} = 3$.

Problem 5. (25 points)

(i) Let a, b be real numbers such that $b \le 0$ and $1 + ax + bx^2 \ge 0$ for every x in $[0, 1]$. Prove that

$$\lim_{n\to+\infty} n\int_0^1 (1 + ax + bx^2)^n dx = \begin{cases} -\dfrac{1}{a} & \text{if } a < 0, \\ +\infty & \text{if } a \ge 0. \end{cases}$$

(ii) Let $f : [0, 1] \to [0, \infty)$ be a function with a continuous second derivative and let $f''(x) \le 0$ for every x in $[0, 1]$. Suppose that $L = \lim_{n\to\infty} n \int_0^1 (f(x))^n\, dx$ exists and $0 < L < +\infty$. Prove that f' has a constant sign and $\min_{x\in[0,1]} |f'(x)| = L^{-1}$.

Solution. (i) With a linear change of the variable (i) is equivalent to:

(i′) Let a, b, A be real numbers such that $b \le 0$, $A > 0$ and $1+ax+bx^2 > 0$ for every x in $[0, A]$. Denote $I_n = n\int_0^A (1 + ax + bx^2)^n dx$. Prove that $\lim_{n\to+\infty} I_n = -\frac{1}{a}$ when $a < 0$ and $\lim_{n\to+\infty} I_n = +\infty$ when $a \ge 0$.

Let $a<0$. Set $f(x)=e^{ax}-(1+ax+bx^2)$. Using that $f(0)=f'(0)=0$ and $f''(x)=a^2e^{ax}-2b$ we get for $x>0$ that

$$0<e^{ax}-(1+ax+bx^2)<cx^2$$

where $c=\dfrac{a^2}{2}-b$. Using the mean value theorem we get

$$0<e^{anx}-(1+ax+bx^2)^n<cx^2ne^{a(n-1)x}.$$

Therefore

$$0<n\int_0^A e^{anx}dx-n\int_0^A(1+ax+bx^2)^n dx<cn^2\int_0^A x^2e^{a(n-1)x}dx.$$

Using that

$$n\int_0^A e^{anx}dx=\frac{e^{anA}-1}{a}\underset{n\to\infty}{\longrightarrow}-\frac{1}{a}$$

and

$$\int_0^A x^2e^{a(n-1)x}dx<\frac{1}{|a|^3(n-1)^3}\int_0^\infty t^2e^{-t}dt$$

we get (i$'$) in the case $a<0$.

Let $a\geq 0$. Then for $n>\max\{A^{-2},-b\}-1$ we have

$$\begin{aligned} n\int_0^A(1+ax+bx^2)^n dx &> n\int_0^{\frac{1}{\sqrt{n+1}}}(1+bx^2)^n dx\\ &> n\cdot\frac{1}{\sqrt{n+1}}\cdot\left(1+\frac{b}{n+1}\right)^n\\ &> \frac{n}{\sqrt{n+1}}e^b\underset{n\to\infty}{\longrightarrow}\infty. \end{aligned}$$

(i) is proved.

(ii) Denote $I_n=n\displaystyle\int_0^1(f(x))^n dx$ and $M=\max\limits_{x\in[0,1]}f(x)$.

For $M<1$ we have $I_n\leq nM^n\underset{n\to\infty}{\longrightarrow}0$, a contradiction.

If $M>1$ since f is continuous there exists an interval $I\subset[0,1]$ with $|I|>0$ such that $f(x)>1$ for every $x\in I$. Then $I_n\geq n|I|\underset{n\to\infty}{\longrightarrow}+\infty$, a contradiction. Hence $M=1$. Now we prove that f' has a constant sign. Assume the opposite. Then $f'(x_0)=0$ for some $x\in(0,1)$. Then

$f(x_0) = M = 1$ because $f'' \leq 0$. For x_0+h in $[0,1]$, $f(x_0+h) = 1+\dfrac{h^2}{2}f''(\xi)$, $\xi \in (x_0, x_0+h)$. Let $m = \min\limits_{x\in[0,1]} f''(x)$. So, $f(x_0+h) \geq 1 + \frac{h^2}{2}m$.

Let $\delta > 0$ be such that $1 + \dfrac{\delta^2}{2}m > 0$ and $x_0 + \delta < 1$. Then

$$I_n \geq n \int_{x_0}^{x_0+\delta} (f(x))^n dx \geq n \int_0^{\delta} \left(1 + \frac{m}{2}h^2\right)^n dh \underset{n\to\infty}{\longrightarrow} \infty$$

in view of (i′) – a contradiction. Hence f is monotone and $M = f(0)$ or $M = f(1)$.

Let $M = f(0) = 1$. For h in $[0,1]$

$$1 + hf'(0) \geq f(h) \geq 1 + hf'(0) + \frac{m}{2}h^2,$$

where $f'(0) \neq 0$, because otherwise we get a contradiction as above. Since $f(0) = M$ the function f is decreasing and hence $f'(0) < 0$. Let $0 < A < 1$ be such that $1 + Af'(0) + \dfrac{m}{2}A^2 > 0$. Then

$$n\int_0^A (1 + hf'(0))^n dh \geq n\int_0^A (f(x))^n dx \geq n\int_0^A \left(1 + hf'(0) + \frac{m}{2}h^2\right)^n dh.$$

From (i′) the first and the third integral tend to $-\dfrac{1}{f'(0)}$ as $n \to \infty$, hence so does the second.

Also $n\displaystyle\int_A^1 (f(x))^n dx \leq n(f(A))^n \underset{n\to\infty}{\longrightarrow} 0$ $(f(A) < 1)$. We get $L = -\dfrac{1}{f'(0)}$ in this case.

If $M = f(1)$ we get in a similar way $L = \dfrac{1}{f'(1)}$.

Problem 6. (25 points)

Upper content of a subset E of the plane $\mathbb{R}^2$ is defined as

$$\mathcal{C}(E) = \inf\left\{\sum_{i=1}^{n} \operatorname{diam}(E_i)\right\}$$

where inf is taken over all finite families of sets $E_1, \ldots, E_n$, $n \in \mathbb{N}$, in $\mathbb{R}^2$ such that $E \subset \bigcup\limits_{i=1}^{n} E_i$.

Lower content of E is defined as

$$\mathcal{K}(E) = \sup\{\text{lenght}(L) \;:\; L \text{ is a closed line segment onto which } E \text{ can be contracted}\}.$$

Show that

(a) $\mathcal{C}(L) = \text{lenght}(L)$ if L is a closed line segment;

(b) $\mathcal{C}(E) \geq \mathcal{K}(E)$;

(c) the equality in (b) needs not hold even if E is compact.

Hint. If $E = T \cup T'$ where T is the triangle with vertices $(-2, 2)$, $(2, 2)$ and $(0, 4)$, and T' is its reflexion about the x-axis, then $\mathcal{C}(E) = 8 > \mathcal{K}(E)$.

Remarks: All *distances* used in this problem are Euclidian. *Diameter* of a set E is $\text{diam}(E) = \sup\{\text{dist}(x, y) \;:\; x, y \in E\}$. *Contraction* of a set E to a set F is a mapping $f : E \mapsto F$ such that $\text{dist}(f(x), f(y)) \leq \text{dist}(x, y)$ for all $x, y \in E$. A set E can be contracted *onto* a set F if there is a contraction f of E to F which is onto, i.e., such that $f(E) = F$. *Triangle* is defined as the union of the three segments joining its vertices, i.e., it does not contain the interior.

Solution.

(a) The choice $E_1 = L$ gives $\mathcal{C}(L) \leq \text{lenght}(L)$. If $E \subset \cup_{i=1}^n E_i$ then $\sum_{i=1}^{n} \text{diam}(E_i) \geq \text{lenght}(L)$: By induction, n=1 obvious, and assuming that E_{n+1} contains the end point a of L, define the segment $L_\varepsilon = \{x \in L \;:\; \text{dist}(x, a) \geq \text{diam}(E_{n+1}) + \varepsilon\}$ and use induction assumption to get $\sum_{i=1}^{n+1} \text{diam}(E_i) \geq \text{lenght}(L_\varepsilon) + \text{diam}(E_{n+1}) \geq \text{lenght}(L) - \varepsilon$; but $\varepsilon > 0$ is arbitrary.

(b) If f is a contraction of E onto L and $E \subset \cup_{n=1}^n E_i$, then $L \subset \cup_{i=1}^n f(E_i)$ and $\text{lenght}(L) \leq \sum_{i=1}^{n} \text{diam}(f(E_i)) \leq \sum_{i=1}^{n} \text{diam}(E_i)$.

(c1) Let $E = T \cup T'$ where T is the triangle with vertices $(-2, 2)$, $(2, 2)$ and $(0, 4)$, and T' is its reflexion about the x-axis. Suppose $E \subset \bigcup_{i=1}^{n} E_i$. If no set among E_i meets both T and T', then E_i may be partitioned into covers of segments $[(-2, 2), (2, 2)]$ and $[(-2, -2), (2, -2)]$, both of length 4, so $\sum_{i=1}^{n} \text{diam}(E_i) \geq 8$. If at least one set among E_i, say E_k, meets both T and T', choose $a \in E_k \cap T$ and $b \in E_k \cap T'$ and note that the sets $E'_i = E_i$ for $i \neq k$, $E'_k = E_k \cup [a, b]$ cover $T \cup T' \cup [a, b]$, which is a set of upper content at least 8, since its orthogonal projection onto y-axis is a segment of length 8. Since $\text{diam}(E_j) = \text{diam}(E'_j)$, we get $\sum_{i=1}^{n} \text{diam}(E_i) \geq 8$.

(c2) Let f be a contraction of E onto $L = [a', b']$. Choose $a = (a_1, a_2)$, $b = (b_1, b_2) \in E$ such that $f(a) = a'$ and $f(b) = b'$. Since $\text{lenght}(L) = \text{dist}(a', b') \leq \text{dist}(a, b)$ and since the triangles have diameter only 4, we may assume that $a \in T$ and $b \in T'$. Observe that if $a_2 \leq 3$ then a lies on one of the segments joining some of the points $(-2, 2)$, $(2, 2)$, $(-1, 3)$, $(1, 3)$; since all these points have distances from vertices, and so from points, of T_2 at most $\sqrt{50}$, we get that $\text{lenght}(L) \leq \text{dist}(a, b) \leq \sqrt{50}$. Similarly if $b_2 \geq -3$. Finally, if $a_2 > 3$ and $b_2 < -3$, we note that every vertex, and so every point of T is in the distance at most $\sqrt{10}$ for a and every vertex, and so every point, of T' is in the distance at most $\sqrt{10}$ of b. Since f is a contraction, the image of T lies in a segment containing a' of length at most $\sqrt{10}$ and the image of T' lies in a segment containing b' of length at most $\sqrt{10}$. Since the union of these two images is L, we get $\text{lenght}(L) \leq 2\sqrt{10} \leq \sqrt{50}$. Thus $\mathcal{K}(E) \leq \sqrt{50} < 8$.

Second day — August 3, 1996

Problem 1. (10 points)

Prove that if $f : [0, 1] \to [0, 1]$ is a continuous function, then the sequence of iterates $x_{n+1} = f(x_n)$ converges if and only if

$$\lim_{n\to\infty} (x_{n+1} - x_n) = 0.$$

Solution. The “only if” part is obvious. Now suppose that $\lim_{n\to\infty} (x_{n+1} - x_n) = 0$ and the sequence $\{x_n\}$ does not converge. Then there are two cluster points $K < L$. There must be points from the interval (K, L) in the sequence. There is an $x \in (K, L)$ such that $f(x) \neq x$. Put $\varepsilon = \dfrac{|f(x) - x|}{2} > 0$. Then from the continuity of the function f we get that for some $\delta > 0$ for all $y \in (x - \delta, x + \delta)$ it is $|f(y) - y| > \varepsilon$. On the other hand for n large enough it is $|x_{n+1} - x_n| < 2\delta$ and $|f(x_n) - x_n| = |x_{n+1} - x_n| < \varepsilon$. So the sequence cannot come into the interval $(x - \delta, x + \delta)$, but also cannot jump over this interval. Then all cluster points have to be at most $x - \delta$ (a contradiction with L being a cluster point), or at least $x + \delta$ (a contradiction with K being a cluster point).

Problem 2. (10 points)

Let θ be a positive real number and let $\cosh t = \dfrac{e^t + e^{-t}}{2}$ denote the hyperbolic cosine. Show that if $k \in \mathbb{N}$ and both $\cosh k\theta$ and $\cosh (k+1)\theta$ are rational, then so is $\cosh \theta$.

Solution. First we show that

(1) If $\cosh t$ is rational and $m \in \mathbb{N}$, then $\cosh mt$ is rational.

Since $\cosh 0.t = \cosh 0 = 1 \in \mathbb{Q}$ and $\cosh 1.t = \cosh t \in \mathbb{Q}$, (1) follows inductively from

$$\cosh (m+1)t = 2\cosh t.\cosh mt - \cosh (m-1)t.$$

The statement of the problem is obvious for $k = 1$, so we consider $k \geq 2$. For any m we have

(2)

$$\begin{aligned}\cosh \theta &= \cosh ((m+1)\theta - m\theta) = \\ &= \cosh (m+1)\theta.\cosh m\theta - \sinh (m+1)\theta.\sinh m\theta \\ &= \cosh (m+1)\theta.\cosh m\theta - \sqrt{\cosh^2 (m+1)\theta - 1}.\sqrt{\cosh^2 m\theta - 1}\end{aligned}$$

Set $\cosh k\theta = a$, $\cosh (k+1)\theta = b$, $a, b \in \mathbb{Q}$. Then (2) with $m = k$ gives

$$\cosh \theta = ab - \sqrt{a^2 - 1}\sqrt{b^2 - 1}$$

and then

$$\begin{aligned}(a^2 - 1)(b^2 - 1) &= (ab - \cosh \theta)^2 \\ &= a^2 b^2 - 2ab\cosh \theta + \cosh^2 \theta.\end{aligned} \tag{3}$$

Set $\cosh (k^2 - 1)\theta = A$, $\cosh k^2\theta = B$. From (1) with $m = k - 1$ and $t = (k+1)\theta$ we have $A \in \mathbb{Q}$. From (1) with $m = k$ and $t = k\theta$ we have $B \in \mathbb{Q}$. Moreover $k^2 - 1 > k$ implies $A > a$ and $B > b$. Thus $AB > ab$. From (2) with $m = k^2 - 1$ we have

$$\begin{aligned}(A^2 - 1)(B^2 - 1) &= (AB - \cosh \theta)^2 \\ &= A^2 B^2 - 2AB\cosh \theta + \cosh^2 \theta.\end{aligned} \tag{4}$$

So after we cancel the $\cosh^2\theta$ from (3) and (4) we have a non-trivial linear equation in $\cosh \theta$ with rational coefficients.

Problem 3. (15 points)
Let G be the subgroup of $GL_2(\mathbb{R})$, generated by A and B, where

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Let H consist of those matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ in G for which $a_{11}{=}a_{22}{=}1$.

(a) Show that H is an abelian subgroup of G.
(b) Show that H is not finitely generated.

Remarks. $GL_2(\mathbb{R})$ denotes, as usual, the group (under matrix multiplication) of all 2×2 invertible matrices with real entries (elements). *Abelian* means commutative. A group is *finitely generated* if there are a finite number of elements of the group such that every other element of the group can be obtained from these elements using the group operation.

Solution.
(a) All of the matrices in G are of the form

$$\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}.$$

So all of the matrices in H are of the form

$$M(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix},$$

so they commute. Since $M(x)^{-1} = M(-x)$, H is a subgroup of G.

(b) A generator of H can only be of the form $M(x)$, where x is a binary rational, i.e., $x = \dfrac{p}{2^n}$ with integer p and non-negative integer n. In H it holds

$$M(x)M(y) = M(x+y)$$
$$M(x)M(y)^{-1} = M(x-y).$$

The matrices of the form $M\left(\dfrac{1}{2^n}\right)$ are in H for all $n \in \mathbb{N}$. With only finite number of generators all of them cannot be achieved.

Problem 4. (20 points)

Let B be a bounded closed convex symmetric (with respect to the origin) set in $\mathbb{R}^2$ with boundary the curve Γ. Let B have the property that the ellipse of maximal area contained in B is the disc D of radius 1 centered at the origin with boundary the circle C. Prove that $A \cap \Gamma \neq \emptyset$ for any arc A of C of length $l(A) \geq \dfrac{\pi}{2}$.

Solution. Assume the contrary – there is an arc $A \subset C$ with length $l(A) = \dfrac{\pi}{2}$ such that $A \subset B \backslash \Gamma$. Without loss of generality we may assume that the ends of A are $M = (1/\sqrt{2}, 1/\sqrt{2})$, $N = (1/\sqrt{2}, -1/\sqrt{2})$. A is compact and Γ is closed. From $A \cap \Gamma = \emptyset$ we get $\delta > 0$ such that $\operatorname{dist}(x, y) > \delta$ for every $x \in A$, $y \in \Gamma$.

Given $\varepsilon > 0$ with E_ε we denote the ellipse with boundary: $\dfrac{x^2}{(1+\varepsilon)^2} + \dfrac{y^2}{b^2} = 1$, such that $M, N \in E_\varepsilon$. Since $M \in E_\varepsilon$ we get

$$b^2 = \frac{(1+\varepsilon)^2}{2(1+\varepsilon)^2 - 1}.$$

Then we have

$$\operatorname{area} E_\varepsilon = \pi \frac{(1+\varepsilon)^2}{\sqrt{2(1+\varepsilon)^2 - 1}} > \pi = \operatorname{area} D.$$

In view of the hypotheses, $E_\varepsilon \setminus B \neq \emptyset$ for every $\varepsilon > 0$. Let $S = \{(x, y) \in \mathbb{R}^2 \ : \ |x| > |y|\}$. ¿From $E_\varepsilon \setminus S \subset D \subset B$ it follows that $E_\varepsilon \setminus B \subset S$. Taking $\varepsilon < \delta$ we get that

$$\emptyset \neq E_\varepsilon \setminus B \subset E_\varepsilon \cap S \subset D_{1+\varepsilon} \cap S \subset B$$

– a contradiction (we use the notation $D_t = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leq t^2\}$).

Remark. The ellipse with maximal area is well known as John's ellipse. Any coincidence with the President of the Jury is accidental.

Problem 5. (20 points)

(i) Prove that

$$\lim_{x \to +\infty} \sum_{n=1}^{\infty} \frac{nx}{(n^2 + x)^2} = \frac{1}{2}.$$

(ii) Prove that there is a positive constant c such that for every $x \in [1, \infty)$ we have

$$\left| \sum_{n=1}^{\infty} \frac{nx}{(n^2 + x)^2} - \frac{1}{2} \right| \leq \frac{c}{x}.$$

Solution.
(i) Set $f(t) = \dfrac{t}{(1+t^2)^2}$, $h = \dfrac{1}{\sqrt{x}}$. Then

$$\sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} = h\sum_{n=1}^{\infty} f(nh) \underset{h\to 0}{\longrightarrow} \int_0^{\infty} f(t)dt = \frac{1}{2}.$$

The convergence holds since $h\sum_{n=1}^{\infty} f(nh)$ is a Riemann sum of the integral $\int_0^{\infty} f(t)dt$. There are no problems with the infinite domain because f is integrable and $f \downarrow 0$ for $x \to \infty$ (thus $h\sum_{n=N}^{\infty} f(nh) \geq \int_{nN}^{\infty} f(t)dt \geq h\sum_{n=N+1}^{\infty} f(nh)$).

(ii) We have

$$\text{(1)} \quad \begin{aligned} \left|\sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} - \frac{1}{2}\right| &= \left|\sum_{n=1}^{\infty} \left(hf(nh) - \int_{nh-\frac{h}{2}}^{nh+\frac{h}{2}} f(t)dt\right) - \int_0^{\frac{h}{2}} f(t)dt\right| \\ &\leq \sum_{n=1}^{\infty} \left|hf(nh) - \int_{nh-\frac{h}{2}}^{nh+\frac{h}{2}} f(t)dt\right| + \int_0^{\frac{h}{2}} f(t)dt \end{aligned}$$

Using twice integration by parts one has

$$\text{(2)} \quad 2bg(a) - \int_{a-b}^{a+b} g(t)dt = -\frac{1}{2}\int_0^b (b-t)^2(g''(a+t) + g''(a-t))dt$$

for every $g \in C^2[a-b, a+b]$. Using $f(0) = 0$, $f \in C^2[0, h/2]$ one gets

$$\text{(3)} \quad \int_0^{h/2} f(t)dt = O(h^2).$$

From (1), (2) and (3) we get

$$\begin{aligned} \left|\sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} - \frac{1}{2}\right| &\leq \sum_{n=1}^{\infty} h^2 \int_{nh-\frac{h}{2}}^{nh+\frac{h}{2}} |f''(t)|dt + O(h^2) = \\ &= h^2 \int_{\frac{h}{2}}^{\infty} |f''(t)|dt + O(h^2) = O(h^2) = O(x^{-1}). \end{aligned}$$

Problem 6. (Carleman's inequality) (25 points)

(i) Prove that for every sequence $\{a_n\}_{n=1}^{\infty}$, such that $a_n > 0$, $n = 1, 2, \dots$ and $\sum_{n=1}^{\infty} a_n < \infty$, we have

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n,$$

where e is the natural log base.

(ii) Prove that for every $\varepsilon > 0$ there exists a sequence $\{a_n\}_{n=1}^{\infty}$, such that $a_n > 0$, $n = 1, 2, \dots$, $\sum_{n=1}^{\infty} a_n < \infty$ and

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} > (e - \varepsilon) \sum_{n=1}^{\infty} a_n.$$

Solution.

(i) Put for $n \in \mathbb{N}$

$$c_n = (n+1)^n / n^{n-1}. \tag{1}$$

Observe that $c_1 c_2 \cdots c_n = (n+1)^n$. Hence, for $n \in \mathbb{N}$,

$$\begin{aligned}(a_1 a_2 \cdots a_n)^{1/n} &= (a_1 c_1 a_2 c_2 \cdots a_n c_n)^{1/n} / (n+1) \\ &\le (a_1 c_1 + \cdots + a_n c_n) / n(n+1).\end{aligned}$$

Consequently,

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} \le \sum_{n=1}^{\infty} a_n c_n \left(\sum_{m=n}^{\infty} (m(m+1))^{-1} \right). \tag{2}$$

Since

$$\sum_{m=n}^{\infty} (m(m+1))^{-1} = \sum_{m=n}^{\infty} \left(\frac{1}{m} - \frac{1}{m+1} \right) = 1/n$$

we have

$$\begin{aligned}\sum_{n=1}^{\infty} a_n c_n \left(\sum_{m=n}^{\infty} (m(m+1))^{-1} \right) &= \sum_{n=1}^{\infty} a_n c_n / n \\ &= \sum_{n=1}^{\infty} a_n ((n+1)/n)^n < e \sum_{n=1}^{\infty} a_n\end{aligned}$$

(by (1)). Combining the last inequality with (2) we get the result.

(ii) Set $a_n = n^{n-1}(n+1)^{-n}$ for $n = 1, 2, \dots, N$ and $a_n = 2^{-n}$ for $n > N$, where N will be chosen later. Then

$$(a_1 \cdots a_n)^{1/n} = \frac{1}{n+1} \tag{3}$$

for $n \le N$. Let $K = K(\varepsilon)$ be such that

$$\left(\frac{n+1}{n}\right)^n > e - \frac{\varepsilon}{2} \quad \text{for} \quad n > K. \tag{4}$$

Choose N from the condition

$$\sum_{n=1}^{K} a_n + \sum_{n=1}^{\infty} 2^{-n} \le \frac{\varepsilon}{(2e-\varepsilon)(e-\varepsilon)} \sum_{n=K+1}^{N} \frac{1}{n}, \tag{5}$$

which is always possible because the harmonic series diverges. Using (3), (4) and (5) we have

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{K} a_n + \sum_{n=N+1}^{\infty} 2^{-n} + \sum_{n=K+1}^{N} \frac{1}{n}\left(\frac{n}{n+1}\right)^n < \\ &< \frac{\varepsilon}{(2e-\varepsilon)(e-\varepsilon)} \sum_{n=K+1}^{N} \frac{1}{n} + \left(e - \frac{\varepsilon}{2}\right)^{-1} \sum_{n=K+1}^{N} \frac{1}{n} = \\ &= \frac{1}{e-\varepsilon} \sum_{n=K+1}^{N} \frac{1}{n} \le \frac{1}{e-\varepsilon} \sum_{n=1}^{\infty} (a_1 \cdots a_n)^{1/n}.\end{aligned}$$