5^{th} INTERNATIONAL MATHEMATICS COMPETITION FOR UNIVERSITY STUDENTS

July 29 - August 3, 1998, Blagoevgrad, Bulgaria

Second day

PROBLEMS AND SOLUTION

Problem 1. *(20 points)* Let V be a real vector space, and let $f, f_1, f_2, \ldots, f_k$ be linear maps from V to $\mathbb{R}$. Suppose that $f(x) = 0$ whenever $f_1(x) = f_2(x) = \ldots = f_k(x) = 0$. Prove that f is a linear combination of f_1, f_2, ..., f_k.

Solution. We use induction on k. By passing to a subset, we may assume that $f_1, \ldots, f_k$ are linearly independent.

Since f_k is independent of $f_1, \ldots, f_{k-1}$, by induction there exists a vector $a_k \in V$ such that $f_1(a_k) = \ldots = f_{k-1}(a_k) = 0$ and $f_k(a_k) \neq 0$. After normalising, we may assume that $f_k(a_k) = 1$. The vectors $a_1, \ldots, a_{k-1}$ are defined similarly to get

$$f_i(a_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

For an arbitrary $x \in V$ and $1 \leq i \leq k$, $f_i(x - f_1(x)a_1 - f_2(x)a_2 - \cdots - f_k(x)a_k) = f_i(x) - \sum_{j=1}^{k} f_j(x) f_i(a_j) = f_i(x) - f_i(x) f_i(a_i) = 0$, thus $f(x - f_1(x)a_1 - \cdots - f_k(x)a_k) = 0$. By the linearity of f this implies $f(x) = f_1(x)f(a_1) + \cdots + f_k(x)f(a_k)$, which gives $f(x)$ as a linear combination of $f_1(x), \ldots, f_k(x)$.

Problem 2. *(20 points)* Let

$$\mathcal{P} = \{f : f(x) = \sum_{k=0}^{3} a_k x^k,\ a_k \in \mathbb{R},\ |f(\pm 1)| \leq 1,\ |f(\pm \frac{1}{2})| \leq 1\}.$$

Evaluate

$$\sup_{f \in \mathcal{P}} \max_{-1 \leq x \leq 1} |f''(x)|$$

and find all polynomials $f \in \mathcal{P}$ for which the above "sup" is attained.

Solution. Denote $x_0 = 1, x_1 = -\frac{1}{2}, x_2 = \frac{1}{2}, x_3 = 1$,

$$w(x) = \prod_{i=0}^{3} (x - x_i),$$

$$w_k(x) = \frac{w(x)}{x - x_k},\ k = 0, \ldots, 3,$$

$$l_k(x) = \frac{w_k(x)}{w_k(x_k)}.$$

Then for every $f \in \mathcal{P}$

$$f''(x) = \sum_{k=0}^{3} l_k''(x) f(x_k),$$

$$|f''(x)| \leq \sum_{k=0}^{3} |l_k''(x)|.$$

Since f'' is a linear function $\max_{-1\le x\le 1}|f''(x)|$ is attained either at $x=-1$ or at $x=1$. Without loss of generality let the maximum point is $x=1$. Then

$$\sup_{f\in\mathcal{P}}\max_{-1\le x\le 1}|f''(x)| = \sum_{k=0}^{3}|l_k''(1)|.$$

In order to have equality for the extremal polynomial f_* there must hold

$$f_*(x_k) = \operatorname{sign} l_k''(1),\ k=0,1,2,3.$$

It is easy to see that $\{l_k''(1)\}_{k=0}^3$ alternate in sign, so $f_*(x_k)=(-1)^{k-1},\ k=0,\dots,3$. Hence $f_*(x) = T_3(x) = 4x^3-3x$, the Chebyshev polynomial of the first kind, and $f_*''(1)=24$. The other extremal polynomial, corresponding to $x=-1$, is $-T_3$.

Problem 3. *(20 points)* Let $0<c<1$ and

$$f(x)=\begin{cases}\frac{x}{c} & \text{for } x\in[0,c],\\ \frac{1-x}{1-c} & \text{for } x\in[c,1].\end{cases}$$

We say that p is an n-periodic point if

$$\underbrace{f(f(\dots f}_{n}(p))) = p$$

and n is the smallest number with this property. Prove that for every $n\ge 1$ the set of n-periodic points is non-empty and finite.

Solution. Let $f_n(x) = \underbrace{f(f(\dots f}_{n}(x)))$. It is easy to see that $f_n(x)$ is a picewise monotone function and its graph contains 2^n linear segments; one endpoint is always on $\{(x,y): 0\le x\le 1,\ y=0\}$, the other is on $\{(x,y): 0\le x\le 1, y=1\}$. Thus the graph of the identity function intersects each segment once, so the number of points for which $f_n(x)=x$ is 2^n.

Since for each n-periodic points we have $f_n(x)=x$, the number of n-periodic points is finite.

A point x is n-periodic if $f_n(x)=x$ but $f_k(x)\ne x$ for $k=1,\dots,n-1$. But as we saw before $f_k(x)=x$ holds only at 2^k points, so there are at most $2^1+2^2+\dots+2^{n-1}=2^n-2$ points x for which $f_k(x)=x$ for at least one $k\in\{1,2,\dots,n-1\}$. Therefore at least two of the 2^n points for which $f_n(x)=x$ are n-periodic points.

Problem 4. *(20 points)* Let $A_n=\{1,2,\dots,n\}$, where $n\ge 3$. Let $\mathcal{F}$ be the family of all non-constant functions $f\colon A_n\to A_n$ satisfying the following conditions:

(1) $f(k)\le f(k+1)$ for $k=1,2,\dots,n-1$,

(2) $f(k)=f(f(k+1))$ for $k=1,2,\dots,n-1$.

Find the number of functions in $\mathcal{F}$.

Solution. It is clear that $id: A_n\longrightarrow A_n$, given by $id(x)=x$, does not verify condition (2). Since id is the only increasing injection on A_n, $\mathcal{F}$ does not contain injections. Let us take any $f\in\mathcal{F}$ and suppose that $\#\left(f^{-1}(k)\right)\ge 2$. Since f is increasing, there exists $i\in A_n$ such that $f(i)=f(i+1)=k$. In view of (2), $f(k)=f\left(f(i+1)\right)=f(i)=k$. If $\{i<k: f(i)<k\}=\emptyset$, then taking $j=\max\{i<k: f(i)<k\}$ we get $f(j)<f(j+1)=k=f\left(f(j+1)\right)$, a contradiction. Hence $f(i)=k$ for $i\le k$. If $\#\left(f^{-1}(\{l\})\right)\ge 2$ for some $l\ge k$, then the similar consideration shows that $f(i)=l=k$ for $i\le k$. Hence $\#\left(f^{-1}\{i\}\right)=0$ or 1 for every $i>k$. Therefore $f(i)\le i$ for $i>k$. If $f(l)=l$, then taking $j=\max\{i<l: f(i)<l\}$ we get $f(j)<f(j+1)=l=f\left(f(j+1)\right)$, a contradiction. Thus, $f(i)\le i-1$ for $i>k$. Let $m=\max\{i: f(i)=k\}$. Since f is non-constant $m\le n-1$. Since $k=f(m)=f\left(f(m+1)\right)$, $f(m+1)\in[k+1,m]$. If $f(l)>l-1$ for some $l>m+1$, then $l-1$ and $f(l)$ belong to $f^{-1}\left(f(l)\right)$ and this contradicts the facts above. Hence $f(i) = i - 1$ for $i > m + 1$. Thus we show that every function f in $\mathcal{F}$ is defined by natural numbers k, l, m, where $1 \leq k < l = f(m+1) \leq m \leq n - 1$.

$$f(i) = \begin{cases} k & \text{if } i \leq m \\ l & \text{if } i = m \\ i - 1 & \text{if } i > m + 1. \end{cases}$$

Then

$$\#(\mathcal{F}) = \begin{pmatrix} n \\ 3 \end{pmatrix}.$$

Problem 5. *(20 points)* Suppose that $\mathcal{S}$ is a family of spheres (i.e., surfaces of balls of positive radius) in $\mathbb{R}^n$, $n \geq 2$, such that the intersection of any two contains at most one point. Prove that the set M of those points that belong to at least two different spheres from $\mathcal{S}$ is countable.

Solution. For every $x \in M$ choose spheres $S, T \in \mathcal{S}$ such that $S \neq T$ and $x \in S \cap T$; denote by U, V, W the three components of $\mathbf{R}^n \setminus (S \cup T)$, where the notation is such that $\partial U = S$, $\partial V = T$ and x is the only point of $\overline{U} \cap \overline{V}$, and choose points with rational coordinates $u \in U$, $v \in V$, and $w \in W$. We claim that x is uniquely determined by the triple $\langle u, v, w \rangle$; since the set of such triples is countable, this will finish the proof.

To prove the claim, suppose, that from some $x' \in M$ we arrived to the same $\langle u, v, w \rangle$ using spheres $S', T' \in \mathcal{S}$ and components U', V', W' of $\mathbf{R}^n \setminus (S' \cup T')$. Since $S \cap S'$ contains at most one point and since $U \cap U' \neq \emptyset$, we have that $U \subset U'$ or $U' \subset U$; similarly for V's and W's. Exchanging the role of x and x' and/or of U's and V's if necessary, there are only two cases to consider: (a) $U \supset U'$ and $V \supset V'$ and (b) $U \subset U'$, $V \supset V'$ and $W \subset W'$. In case (a) we recall that $\overline{U} \cap \overline{V}$ contains only x and that $x' \in \overline{U'} \cap \overline{V'}$, so $x = x'$. In case (b) we get from $W \subset W'$ that $U' \subset \overline{U \cup V}$; so since U' is open and connected, and $\overline{U} \cap \overline{V}$ is just one point, we infer that $U' = U$ and we are back in the already proved case (a).

Problem 6. *(20 points)* Let $f\colon (0,1) \to [0, \infty)$ be a function that is zero except at the distinct points $a_1, a_2, \ldots$. Let $b_n = f(a_n)$.

(a) Prove that if $\sum_{n=1}^{\infty} b_n < \infty$, then f is differentiable at at least one point $x \in (0,1)$.

(b) Prove that for any sequence of non-negative real numbers $(b_n)_{n=1}^{\infty}$, with $\sum_{n=1}^{\infty} b_n = \infty$, there exists a sequence $(a_n)_{n=1}^{\infty}$ such that the function f defined as above is nowhere differentiable.

Solution

a) We first construct a sequence c_n of positive numbers such that $c_n \to \infty$ and $\sum_{n=1}^{\infty} c_n b_n < \frac{1}{2}$. Let $B = \sum_{n=1}^{\infty} b_n$, and for each $k = 0, 1, \ldots$ denote by N_k the first positive integer for which

$$\sum_{n=N_k}^{\infty} b_n \leq \frac{B}{4^k}.$$

Now set $c_n = \frac{2^k}{5B}$ for each n, $N_k \leq n < N_{k+1}$. Then we have $c_n \to \infty$ and

$$\sum_{n=1}^{\infty} c_n b_n = \sum_{k=0}^{\infty} \sum_{N_k \leq n < N_{k+1}} c_n b_n \leq \sum_{k=0}^{\infty} \frac{2^k}{5B} \sum_{n=N_k}^{\infty} b_n \leq \sum_{k=0}^{\infty} \frac{2^k}{5B} \cdot \frac{B}{4^k} = \frac{2}{5}.$$

Consider the intervals $I_n = (a_n - c_n b_n, a_n + c_n b_n)$. The sum of their lengths is $2 \sum c_n b_n < 1$, thus there exists a point $x_0 \in (0,1)$ which is not contained in any I_n. We show that f is differentiable at x_0, and $f'(x_0)=0$. Since x_0 is outside of the intervals I_n, $x_0 \neq a_n$ for any n and $f(x_0)=0$. For arbitrary $x \in (0,1) \setminus \{x_0\}$, if $x = a_n$ for some n, then

$$\left|\frac{f(x)-f(x_0)}{x-x_0}\right| = \frac{f(a_n)-0}{|a_n-x_0|} \leq \frac{b_n}{c_n b_n} = \frac{1}{c_n},$$

otherwise $\frac{f(x)-f(x_0)}{x-x_0}=0$. Since $c_n \to \infty$, this implies that for arbitrary $\varepsilon > 0$ there are only finitely many $x \in (0,1) \setminus \{x_0\}$ for which

$$\left|\frac{f(x)-f(x_0)}{x-x_0}\right| < \varepsilon$$

does not hold, and we are done.

Remark. The variation of f is finite, which implies that f is differentiable almost everywhere .

b) We remove the zero elements from sequence b_n. Since $f(x)=0$ except for a countable subset of $(0,1)$, if f is differentiable at some point x_0, then $f(x_0)$ and $f'(x_0)$ must be 0.

It is easy to construct a sequence β_n satisfying $0 < \beta_n \leq b_n$, $b_n \to 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$.

Choose the numbers $a_1, a_2, \dots$ such that the intervals $I_n = (a_n - \beta_n,\, a_n + \beta_n)$ $(n = 1, 2, \dots)$ cover each point of (0, 1) infinitely many times (it is possible since the sum of lengths is $2\sum b_n = \infty$). Then for arbitrary $x_0 \in (0,\,1)$, $f(x_0)=0$ and $\varepsilon > 0$ there is an n for which $\beta_n < \varepsilon$ and $x_0 \in I_n$ which implies

$$\frac{|f(a_n)-f(x_0)|}{|a_n-x_0|} > \frac{b_n}{\beta_n} \geq 1.$$