Solutions for the first day problems at the IMC 2000

Problem 1.

Is it true that if $f : [0,1] \to [0,1]$ *is*
a) monotone increasing
b) monotone decreasing
then there exists an $x \in [0,1]$ *for which* $f(x) = x$*?*

Solution.

a) Yes.

Proof: Let $A = \{x \in [0,1] : f(x) > x\}$. If $f(0) = 0$ we are done, if not then A is non-empty (0 is in A) bounded, so it has supremum, say a. Let $b = f(a)$.

I. case: $a < b$. Then, using that f is monotone and a was the sup, we get $b = f(a) \le f((a+b)/2) \le (a+b)/2$, which contradicts $a < b$.

II. case: $a > b$. Then we get $b = f(a) \ge f((a+b)/2) > (a+b)/2$ contradiction. Therefore we must have $a = b$.

b) No. Let, for example,

$$f(x) = 1 - x/2 \quad \text{if} \quad x \le 1/2$$

and

$$f(x) = 1/2 - x/2 \quad \text{if} \quad x > 1/2$$

This is clearly a good counter-example.

Problem 2.

Let $p(x) = x^5 + x$ *and* $q(x) = x^5 + x^2$*. Find all pairs* (w, z) *of complex numbers with* $w \ne z$ *for which* $p(w) = p(z)$ *and* $q(w) = q(z)$*.*

Short solution. Let

$$P(x, y) = \frac{p(x) - p(y)}{x - y} = x^4 + x^3y + x^2y^2 + xy^3 + y^4 + 1$$

and

$$Q(x, y) = \frac{q(x) - q(y)}{x - y} = x^4 + x^3y + x^2y^2 + xy^3 + y^4 + x + y.$$

We need those pairs (w, z) which satisfy $P(w, z) = Q(w, z) = 0$.

From $P - Q = 0$ we have $w + z = 1$. Let $c = wz$. After a short calculation we obtain $c^2 - 3c + 2 = 0$, which has the solutions $c = 1$ and $c = 2$. From the system $w + z = 1$, $wz = c$ we obtain the following pairs:

$$\left(\frac{1 \pm \sqrt{3}i}{2}, \frac{1 \mp \sqrt{3}i}{2} \right) \quad \text{and} \quad \left(\frac{1 \pm \sqrt{7}i}{2}, \frac{1 \mp \sqrt{7}i}{2} \right).$$

Problem 3.
A and B are square complex matrices of the same size and

$$\mathrm{rank}(AB - BA) = 1\,.$$

Show that $(AB - BA)^2 = 0$.

Let $C = AB - BA$. Since $\mathrm{rank}\, C = 1$, at most one eigenvalue of C is different from 0. Also $\mathrm{tr}\, C = 0$, so all the eigevalues are zero. In the Jordan canonical form there can only be one 2×2 cage and thus $C^2 = 0$.

Problem 4.
a) Show that if (x_i) *is a decreasing sequence of positive numbers then*

$$\left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \le \sum_{i=1}^{n} \frac{x_i}{\sqrt{i}}.$$

b) Show that there is a constant C *so that if* (x_i) *is a decreasing sequence of positive numbers then*

$$\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \left(\sum_{i=m}^{\infty} x_i^2\right)^{1/2} \le C \sum_{i=1}^{\infty} x_i.$$

Solution.

a)

$$(\sum_{i=1}^{n} \frac{x_i}{\sqrt{i}})^2 = \sum_{i,j}^{n} \frac{x_i x_j}{\sqrt{i}\sqrt{j}} \ge \sum_{i=1}^{n} \frac{x_i}{\sqrt{i}} \sum_{j=1}^{i} \frac{x_i}{\sqrt{j}} \ge \sum_{i=1}^{n} \frac{x_i}{\sqrt{i}} i \frac{x_i}{\sqrt{i}} = \sum_{i=1}^{n} x_i^2$$

b)

$$\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} (\sum_{i=m}^{\infty} x_i^2)^{1/2} \le \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \sum_{i=m}^{\infty} \frac{x_i}{\sqrt{i - m + 1}}$$

by a)

$$= \sum_{i=1}^{\infty} x_i \sum_{m=1}^{i} \frac{1}{\sqrt{m}\sqrt{i - m + 1}}$$

You can get a sharp bound on

$$\sup_i \sum_{m=1}^{i} \frac{1}{\sqrt{m}\sqrt{i - m + 1}}$$

by checking that it is at most

$$\int_0^{i+1} \frac{1}{\sqrt{x}\sqrt{i + 1 - x}} dx = \pi$$

Alternatively you can observe that

$$\sum_{m=1}^{i} \frac{1}{\sqrt{m}\sqrt{i+1-m}} = 2\sum_{m=1}^{i/2} \frac{1}{\sqrt{m}\sqrt{i+1-m}} \le$$

$$\le 2\frac{1}{\sqrt{i/2}} \sum_{m=1}^{i/2} \frac{1}{\sqrt{m}} \le 2\frac{1}{\sqrt{i/2}}.2\sqrt{i/2} = 4$$

Problem 5.

Let R be a ring of characteristic zero (not necessarily commutative). Let e, f and g be idempotent elements of R satisfying $e + f + g = 0$. Show that $e = f = g = 0$.

(R is of characteristic zero means that, if $a \in R$ and n is a positive integer, then $na \neq 0$ unless $a = 0$. An idempotent x is an element satisfying $x = x^2$.)

Solution. Suppose that $e + f + g = 0$ for given idempotents $e, f, g \in R$. Then

$$g = g^2 = (-(e+f))^2 = e + (ef + fe) + f = (ef + fe) - g,$$

i.e. ef+fe=2g, whence the additive commutator

$$[e, f] = ef - fe = [e, ef + fe] = 2[e, g] = 2[e, -e - f] = -2[e, f],$$

i.e. $ef = fe$ (since R has zero characteristic). Thus $ef + fe = 2g$ becomes $ef = g$, so that $e + f + ef = 0$. On multiplying by e, this yields $e + 2ef = 0$, and similarly $f + 2ef = 0$, so that $f = -2ef = e$, hence $e = f = g$ by symmetry. Hence, finaly, $3e = e + f + g = 0$, i.e. $e = f = g = 0$.

For part (i) just omit some of this.

Problem 6.

Let $f : \mathbb{R} \to (0, \infty)$ be an increasing differentiable function for which $\lim\limits_{x\to\infty} f(x) = \infty$ and f' is bounded.

Let $F(x) = \int\limits_0^x f$. Define the sequence (a_n) inductively by

$$a_0 = 1, \quad a_{n+1} = a_n + \frac{1}{f(a_n)},$$

and the sequence (b_n) simply by $b_n = F^{-1}(n)$. Prove that $\lim\limits_{n\to\infty} (a_n - b_n) = 0$.

Solution. From the conditions it is obvious that F is increasing and $\lim\limits_{n\to\infty} b_n = \infty$.

By Lagrange's theorem and the recursion in (1), for all $k \ge 0$ integers there exists a real number $\xi \in (a_k, a_{k+1})$ such that

$$F(a_{k+1}) - F(a_k) = f(\xi)(a_{k+1} - a_k) = \frac{f(\xi)}{f(a_k)}. \tag{2}$$

By the monotonity, $f(a_k) \le f(\xi) \le f(a_{k+1})$, thus

$$1 \le F(a_{k+1}) - F(a_k) \le \frac{f(a_{k+1})}{f(a_k)} = 1 + \frac{f(a_{k+1}) - f(a_k)}{f(a_k)}. \tag{3}$$

Summing (3) for $k = 0, \dots, n-1$ and substituting $F(b_n) = n$, we have

$$F(b_n) < n + F(a_0) \le F(a_n) \le F(b_n) + F(a_0) + \sum_{k=0}^{n-1} \frac{f(a_{k+1}) - f(a_k)}{f(a_k)}. \tag{4}$$

From the first two inequalities we already have $a_n > b_n$ and $\lim\limits_{n\to\infty} a_n = \infty$.

Let ε be an arbitrary positive number. Choose an integer K_ε such that $f(a_{K_\varepsilon}) > \frac{2}{\varepsilon}$. If n is sufficiently large, then

$$F(a_0) + \sum_{k=0}^{n-1} \frac{f(a_{k+1}) - f(a_k)}{f(a_k)} =$$

$$= \left(F(a_0) + \sum_{k=0}^{K_\varepsilon - 1} \frac{f(a_{k+1}) - f(a_k)}{f(a_k)} \right) + \sum_{k=K_\varepsilon}^{n-1} \frac{f(a_{k+1}) - f(a_k)}{f(a_k)} < \tag{5}$$

$$< O_\varepsilon(1) + \frac{1}{f(a_{K_\varepsilon})} \sum_{k=K_\varepsilon}^{n-1} \big(f(a_{k+1}) - f(a_k) \big) <$$

$$< O_\varepsilon(1) + \frac{\varepsilon}{2} \big(f(a_n) - f(a_{K_\varepsilon}) \big) < \varepsilon f(a_n).$$

Inequalities (4) and (5) together say that for any positive ε, if n is sufficiently large,

$$F(a_n) - F(b_n) < \varepsilon f(a_n).$$

Again, by Lagrange's theorem, there is a real number $\zeta \in (b_n, a_n)$ such that

$$F(a_n) - F(b_n) = f(\zeta)(a_n - b_n) > f(b_n)(a_n - b_n), \tag{6}$$

thus

$$f(b_n)(a_n - b_n) < \varepsilon f(a_n). \tag{7}$$

Let B be an upper bound for f'. Apply $f(a_n) < f(b_n) + B(a_n - b_n)$ in (7):

$$f(b_n)(a_n - b_n) < \varepsilon \big(f(b_n) + B(a_n - b_n) \big),$$

$$\big(f(b_n) - \varepsilon B \big)(a_n - b_n) < \varepsilon f(b_n). \tag{8}$$

Due to $\lim\limits_{n\to\infty} f(b_n) = \infty$, the first factor is positive, and we have

$$a_n - b_n < \varepsilon \frac{f(b_n)}{f(b_n) - \varepsilon B} < 2\varepsilon \tag{9}$$

for sufficiently large n.

Thus, for arbitrary positive ε we proved that $0 < a_n - b_n < 2\varepsilon$ if n is sufficiently large.