12^{th} International Mathematics Competition for University Students

Blagoevgrad, July 22 - July 28, 2005

First Day

Problem 1. Let A be the $n \times n$ matrix, whose $(i,j)^{\text{th}}$ entry is $i+j$ for all $i,j = 1,2,\dots,n$. What is the rank of A?

Solution 1. For $n=1$ the rank is 1. Now assume $n \ge 2$. Since $A = (i)_{i,j=1}^n + (j)_{i,j=1}^n$, matrix A is the sum of two matrixes of rank 1. Therefore, the rank of A is at most 2. The determinant of the top-left 2×2 minor is -1, so the rank is exactly 2.

Therefore, the rank of A is 1 for $n=1$ and 2 for $n \ge 2$.

Solution 2. Consider the case $n \ge 2$. For $i = n, n-1, \dots, 2$, subtract the $(i-1)^{\text{th}}$ row from the n^{th} row. Then subtract the second row from all lower rows.

$$rank \begin{pmatrix} 2 & 3 & \dots & n+1 \\ 3 & 4 & \dots & n+2 \\ \vdots & & \ddots & \vdots \\ n+1 & n+2 & \dots & 2n \end{pmatrix} = rank \begin{pmatrix} 2 & 3 & \dots & n+1 \\ 1 & 1 & \dots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} = rank \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} = 2.$$

Problem 2. For an integer $n \ge 3$ consider the sets

$$S_n = \{(x_1, x_2, \dots, x_n) : \forall i \; x_i \in \{0,1,2\}\}$$

$$A_n = \{(x_1, x_2, \dots, x_n) \in S_n : \forall i \le n-2 \; |\{x_i, x_{i+1}, x_{i+2}\}| \ne 1\}$$

and

$$B_n = \{(x_1, x_2, \dots, x_n) \in S_n : \forall i \le n-1 \; (x_i = x_{i+1} \Rightarrow x_i \ne 0)\}.$$

Prove that $|A_{n+1}| = 3 \cdot |B_n|$.

($|A|$ denotes the number of elements of the set A.)

Solution 1. Extend the definitions also for $n = 1, 2$. Consider the following sets

$$A'_n = \{(x_1, x_2, \dots, x_n) \in A_n : x_{n-1} = x_n\}, \quad A''_n = A_n \setminus A'_n,$$

$$B'_n = \{(x_1, x_2, \dots, x_n) \in B_n : x_n = 0\}, \quad B''_n = B_n \setminus B'_n$$

and denote $a_n = |A_n|$, $a'_n = |A'_n|$, $a''_n = |A''_n|$, $b_n = |B_n|$, $b'_n = |B'_n|$, $b''_n = |B''_n|$.

It is easy to observe the following relations between the a–sequences

$$\begin{cases} a_n &= a'_n + a''_n \\ a'_{n+1} &= a''_n \\ a''_{n+1} &= 2a'_n + 2a''_n \end{cases},$$

which lead to $a_{n+1} = 2a_n + 2a_{n-1}$.

For the b–sequences we have the same relations

$$\begin{cases} b_n &= b'_n + b''_n \\ b'_{n+1} &= b''_n \\ b''_{n+1} &= 2b'_n + 2b''_n \end{cases},$$

therefore $b_{n+1} = 2b_n + 2b_{n-1}$.

By computing the first values of (a_n) and (b_n) we obtain

$$\begin{cases} a_1 = 3, \quad a_2 = 9, \quad a_3 = 24 \\ \qquad\quad b_1 = 3, \quad b_2 = 8 \end{cases}$$

which leads to

$$\begin{cases} a_2 = 3b_1 \\ a_3 = 3b_2 \end{cases}$$

Now, reasoning by induction, it is easy to prove that $a_{n+1} = 3b_n$ for every $n \geq 1$.
Solution 2. Regarding x_i to be elements of $\mathbb{Z}_3$ and working "modulo 3", we have that

$$(x_1, x_2, \ldots, x_n) \in A_n \Rightarrow (x_1 + 1, x_2 + 1, \ldots, x_n + 1) \in A_n, \ (x_1 + 2, x_2 + 2, \ldots, x_n + 2) \in A_n$$

which means that 1/3 of the elements of A_n start with 0. We establish a bijection between the subset of all the vectors in A_{n+1} which start with 0 and the set B_n by

$$(0, x_1, x_2, \ldots, x_n) \in A_{n+1} \longmapsto (y_1, y_2, \ldots, y_n) \in B_n$$
$$y_1 = x_1, y_2 = x_2 - x_1, y_3 = x_3 - x_2, \ldots, y_n = x_n - x_{n-1}$$

(if $y_k = y_{k+1} = 0$ then $x_k - x_{k-1} = x_{k+1} - x_k = 0$ (where $x_0 = 0$), which gives $x_{k-1} = x_k = x_{k+1}$, which is not possible because of the definition of the sets A_p; therefore, the definition of the above function is correct).

The inverse is defined by

$$(y_1, y_2, \ldots, y_n) \in B_n \longmapsto (0, x_1, x_2, \ldots, x_n) \in A_{n+1}$$
$$x_1 = y_1, x_2 = y_1 + y_2, \ldots, x_n = y_1 + y_2 + \cdots + y_n$$

Problem 3. Let $f : \mathbb{R} \to [0, \infty)$ be a continuously differentiable function. Prove that

$$\left| \int_0^1 f^3(x)\, dx - f^2(0) \int_0^1 f(x)\, dx \right| \leq \max_{0 \leq x \leq 1} |f'(x)| \left(\int_0^1 f(x)\, dx \right)^2.$$

Solution 1. Let $M = \max\limits_{0 \leq x \leq 1} |f'(x)|$. By the inequality $-M \leq f'(x) \leq M$, $x \in [0,1]$ it follows:

$$-Mf(x) \leq f(x) f'(x) \leq Mf(x), \ x \in [0,1].$$

By integration

$$-M \int_0^x f(t)\, dt \leq \frac{1}{2} f^2(x) - \frac{1}{2} f^2(0) \leq M \int_0^x f(t)\, dt, \ x \in [0,1]$$
$$-Mf(x) \int_0^x f(t)\, dt \leq \frac{1}{2} f^3(x) - \frac{1}{2} f^2(0) f(x) \leq Mf(x) \int_0^x f(t)\, dt, \ x \in [0,1].$$

Integrating the last inequality on $[0,1]$ it follows that

$$-M \left(\int_0^1 f(x) dx \right)^2 \leq \int_0^1 f^3(x)\, dx - f^2(0) \int_0^1 f(x)\, dx \leq M \left(\int_0^1 f(x) dx \right)^2 \Leftrightarrow$$
$$\left| \int_0^1 f^3(x)\, dx - f^2(0) \int_0^1 f(x)\, dx \right| \leq M \left(\int_0^1 f(x)\, dx \right)^2.$$

Solution 2. Let $M = \max\limits_{0 \leq x \leq 1} |f'(x)|$ and $F(x) = -\int_x^1 f$; then $F' = f$, $F(0) = -\int_0^1 f$ and $F(1) = 0$. Integrating by parts,

$$\int_0^1 f^3 = \int_0^1 f^2 \cdot F' = [f^2 F]_0^1 - \int_0^1 (f^2)' F =$$
$$= f^2(1) F(1) - f^2(0) F(0) - \int_0^1 2Fff' = f^2(0) \int_0^1 f - \int_0^1 2Fff'.$$

Then

$$\left| \int_0^1 f^3(x)\, dx - f^2(0) \int_0^1 f(x)\, dx \right| = \left| \int_0^1 2Fff' \right| \leq \int_0^1 2Ff|f'| \leq M \int_0^1 2Ff = M \cdot [F^2]_0^1 = M \left(\int_0^1 f \right)^2.$$

Problem 4. Find all polynomials $P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ $(a_n \neq 0)$ satisfying the following two conditions:

(i) $(a_0, a_1, ..., a_n)$ is a permutation of the numbers $(0, 1, ..., n)$

and

(ii) all roots of $P(x)$ are rational numbers.

Solution 1. Note that $P(x)$ does not have any positive root because $P(x) > 0$ for every $x > 0$. Thus, we can represent them in the form $-\alpha_i$, $i = 1, 2, \ldots, n$, where $\alpha_i \geq 0$. If $a_0 \neq 0$ then there is a $k \in \mathbb{N}, 1 \leq k \leq n-1$, with $a_k = 0$, so using Viete's formulae we get

$$\alpha_1\alpha_2...\alpha_{n-k-1}\alpha_{n-k} + \alpha_1\alpha_2...\alpha_{n-k-1}\alpha_{n-k+1} + ... + \alpha_{k+1}\alpha_{k+2}...\alpha_{n-1}\alpha_n = \frac{a_k}{a_n} = 0,$$

which is impossible because the left side of the equality is positive. Therefore $a_0 = 0$ and one of the roots of the polynomial, say α_n, must be equal to zero. Consider the polynomial $Q(x) = a_n x^{n-1} + a_{n-1}x^{n-2} + ... + a_1$. It has zeros $-\alpha_i$, $i = 1, 2, \ldots, n-1$. Again, Viete's formulae, for $n \geq 3$, yield:

$$\alpha_1\alpha_2...\alpha_{n-1} = \frac{a_1}{a_n} \tag{1}$$

$$\alpha_1\alpha_2...\alpha_{n-2} + \alpha_1\alpha_2...\alpha_{n-3}\alpha_{n-1} + ... + \alpha_2\alpha_3...\alpha_{n-1} = \frac{a_2}{a_n} \tag{2}$$

$$\alpha_1 + \alpha_2 + ... + \alpha_{n-1} = \frac{a_{n-1}}{a_n}. \tag{3}$$

Dividing (2) by (1) we get

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + ... + \frac{1}{\alpha_{n-1}} = \frac{a_2}{a_1}. \tag{4}$$

From (3) and (4), applying the AM-HM inequality we obtain

$$\frac{a_{n-1}}{(n-1)a_n} = \frac{\alpha_1 + \alpha_2 + ... + \alpha_{n-1}}{n-1} \geq \frac{n-1}{\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + ... + \frac{1}{\alpha_{n-1}}} = \frac{(n-1)a_1}{a_2},$$

therefore $\frac{a_2 a_{n-1}}{a_1 a_n} \geq (n-1)^2$. Hence $\frac{n^2}{2} \geq \frac{a_2 a_{n-1}}{a_1 a_n} \geq (n-1)^2$, implying $n \leq 3$. So, the only polynomials possibly satisfying (i) and (ii) are those of degree at most three. These polynomials can easily be found and they are $P(x) = x$, $P(x) = x^2 + 2x$, $P(x) = 2x^2 + x$, $P(x) = x^3 + 3x^2 + 2x$ and $P(x) = 2x^3 + 3x^2 + x$. □

Solution 2. Consider the prime factorization of P in the ring $\mathbb{Z}[x]$. Since all roots of P are rational, P can be written as a product of n linear polynomials with rational coefficients. Therefore, all prime factor of P are linear and P can be written as

$$P(x) = \prod_{k=1}^{n}(b_k x + c_k)$$

where the coefficients b_k, c_k are integers. Since the leading coefficient of P is positive, we can assume $b_k > 0$ for all k. The coefficients of P are nonnegative, so P cannot have a positive root. This implies $c_k \geq 0$. It is not possible that $c_k = 0$ for two different values of k, because it would imply $a_0 = a_1 = 0$. So $c_k > 0$ in at least $n-1$ cases.

Now substitute $x = 1$.

$$P(1) = a_n + \cdots + a_0 = 0 + 1 + \cdots + n = \frac{n(n+1)}{2} = \prod_{k=1}^{n}(b_k + c_k) \geq 2^{n-1};$$

therefore it is necessary that $2^{n-1} \leq \frac{n(n+1)}{2}$, therefore $n \leq 4$. Moreover, the number $\frac{n(n+1)}{2}$ can be written as a product of $n-1$ integers greater than 1.

If $n = 1$, the only solution is $P(x) = 1x + 0$.

If $n = 2$, we have $P(1) = 3 = 1 \cdot 3$, so one factor must be x, the other one is $x + 2$ or $2x + 1$. Both $x(x+2) = 1x^2 + 2x + 0$ and $x(2x+1) = 2x^2 + 1x + 0$ are solutions.

If $n=3$, then $P(1)=6=1\cdot2\cdot3$, so one factor must be x, another one is $x+1$, the third one is again $x+2$ or $2x+1$. The two polynomials are $x(x+1)(x+2)=1x^3+3x^2+2x+0$ and $x(x+1)(2x+1)=2x^3+3x^2+1x+0$, both have the proper set of coefficients.

In the case $n=4$, there is no solution because $\frac{n(n+1)}{2}=10$ cannot be written as a product of 3 integers greater than 1.

Altogether we found 5 solutions: $1x+0$, $1x^2+2x+0$, $2x^2+1x+0$, $1x^3+3x^2+2x+0$ and $2x^3+3x^2+1x+0$.

Problem 5. Let $f:\ (0,\infty)\to\mathbb{R}$ be a twice continuously differentiable function such that

$$|f''(x)+2xf'(x)+(x^2+1)f(x)|\le 1$$

for all x. Prove that $\lim\limits_{x\to\infty} f(x)=0$.

Solution 1. Let $g(x)=f'(x)+xf(x)$; then $f''(x)+2xf'(x)+(x^2+1)f(x)=g'(x)+xg(x)$.

We prove that if h is a continuously differentiable function such that $h'(x)+xh(x)$ is bounded then $\lim\limits_{\infty} h=0$. Applying this lemma for $h=g$ then for $h=f$, the statement follows.

Let M be an upper bound for $|h'(x)+xh(x)|$ and let $p(x)=h(x)e^{x^2/2}$. (The function $e^{-x^2/2}$ is a solution of the differential equation $u'(x)+xu(x)=0$.) Then

$$|p'(x)|=|h'(x)+xh(x)|e^{x^2/2}\le Me^{x^2/2}$$

and

$$|h(x)|=\left|\frac{p(x)}{e^{x^2/2}}\right|=\left|\frac{p(0)+\int_0^x p'}{e^{x^2/2}}\right|\le\frac{|p(0)|+M\int_0^x e^{x^2/2}dx}{e^{x^2/2}}.$$

Since $\lim\limits_{x\to\infty} e^{x^2/2}=\infty$ and $\lim\dfrac{\int_0^x e^{x^2/2}dx}{e^{x^2/2}}=0$ (by L'Hospital's rule), this implies $\lim_{x\to\infty}h(x)=0$.

Solution 2. Apply L'Hospital rule twice on the fraction $\dfrac{f(x)e^{x^2/2}}{e^{x^2/2}}$. (Note that L'Hospital rule is valid if the denominator converges to infinity, without any assumption on the numerator.)

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{f(x)e^{x^2/2}}{e^{x^2/2}}=\lim_{x\to\infty}\frac{(f'(x)+xf(x))e^{x^2/2}}{xe^{x^2/2}}=\lim_{x\to\infty}\frac{(f''(x)+2xf'(x)+(x^2+1)f(x))e^{x^2/2}}{(x^2+1)e^{x^2/2}}=$$

$$=\lim_{x\to\infty}\frac{f''(x)+2xf'(x)+(x^2+1)f(x)}{x^2+1}=0.$$

Problem 6. Given a group G, denote by $G(m)$ the subgroup generated by the m^{th} powers of elements of G. If $G(m)$ and $G(n)$ are commutative, prove that $G(\gcd(m,n))$ is also commutative. ($\gcd(m,n)$ denotes the greatest common divisor of m and n.)

Solution. Write $d=\gcd(m,n)$. It is easy to see that $\langle G(m),G(n)\rangle=G(d)$; hence, it will suffice to check commutativity for any two elements in $G(m)\cup G(n)$, and so for any two generators a^m and b^n. Consider their commutator $z=a^{-m}b^{-n}a^mb^n$; then the relations

$$z=(a^{-m}ba^m)^{-n}b^n=a^{-m}(b^{-n}ab^n)^m$$

show that $z\in G(m)\cap G(n)$. But then z is in the center of $G(d)$. Now, from the relation $a^mb^n=b^na^mz$, it easily follows by induction that

$$a^{ml}b^{nl}=b^{nl}a^{ml}z^{l^2}.$$

Setting $l=m/d$ and $l=n/d$ we obtain $z^{(m/d)^2}=z^{(n/d)^2}=e$, but this implies that $z=e$ as well.