12^{th} International Mathematics Competition for University Students

Blagoevgrad, July 22 - July 28, 2005

Second Day

Problem 1. Let $f(x) = x^2 + bx + c$, where b and c are real numbers, and let

$$M = \{x \in \mathbb{R} : \ |f(x)| < 1\}.$$

Clearly the set M is either empty or consists of disjoint open intervals. Denote the sum of their lengths by $|M|$. Prove that

$$|M| \le 2\sqrt{2}.$$

Solution. Write $f(x) = \left(x + \frac{b}{2}\right)^2 + d$ where $d = c - \frac{b^2}{4}$. The absolute minimum of f is d.

If $d \ge 1$ then $f(x) \ge 1$ for all x, $M = \emptyset$ and $|M| = 0$.

If $-1 < d < 1$ then $f(x) > -1$ for all x,

$$-1 < \left(x + \frac{b}{2}\right)^2 + d < 1 \quad \Longleftrightarrow \quad \left|x + \frac{b}{2}\right| < \sqrt{1-d}$$

so

$$M = \left(-\frac{b}{2} - \sqrt{1-d}, -\frac{b}{2} + \sqrt{1-d}\right)$$

and

$$|M| = 2\sqrt{1-d} < 2\sqrt{2}.$$

If $d \le -1$ then

$$-1 < \left(x + \frac{b}{2}\right)^2 + d < 1 \quad \Longleftrightarrow \quad \sqrt{|d|-1} < \left|x + \frac{b}{2}\right| < \sqrt{|d|+1}$$

so

$$M = \big(-\sqrt{|d|+1}, -\sqrt{|d|-1}\big) \cup \big(\sqrt{|d|-1}, \sqrt{|d|+1}\big)$$

and

$$|M| = 2\left(\sqrt{|d|+1} - \sqrt{|d|-1}\right) = 2\frac{(|d|+1) - (|d|-1)}{\sqrt{|d|+1} + \sqrt{|d|-1}} \le 2\frac{2}{\sqrt{1+1} + \sqrt{1-0}} = 2\sqrt{2}.$$

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $(f(x))^n$ is a polynomial for every $n = 2, 3, \ldots$. Does it follow that f is a polynomial?

Solution 1. Yes, it is even enough to assume that f^2 and f^3 are polynomials.

Let $p = f^2$ and $q = f^3$. Write these polynomials in the form of

$$p = a \cdot p_1^{a_1} \cdot \ldots \cdot p_k^{a_k}, \qquad q = b \cdot q_1^{b_1} \cdot \ldots \cdot q_l^{b_l},$$

where $a, b \in \mathbb{R}$, $a_1, \dots, a_k, b_1, \dots b_l$ are positive integers and $p_1, \dots, p_k, q_1, \dots, q_l$ are irreducible polynomials with leading coefficients 1. For $p^3 = q^2$ and the factorisation of $p^3 = q^2$ is unique we get that $a^3 = b^2$, $k = l$ and for some $(i_1, \dots, i_k)$ permutation of $(1, \dots, k)$ we have $p_1 = q_{i_1}, \dots, p_k = q_{i_k}$ and $3a_1 = 2b_{i_1}, \dots, 3a_k = 2b_{i_k}$. Hence $b_1, \dots, b_l$ are divisible by 3 let $r = b^{1/3} \cdot q_1^{b_1/3} \cdot \dots \cdot q_l^{b_l/3}$ be a polynomial. Since $r^3 = q = f^3$ we have $f = r$.

Solution 2. Let $\frac{p}{q}$ be the simplest form of the rational function $\frac{f^3}{f^2}$. Then the simplest form of its square is $\frac{p^2}{q^2}$. On the other hand $\frac{p^2}{q^2} = \left(\frac{f^3}{f^2}\right)^2 = f^2$ is a polynomial therefore q must be a constant and so $f = \frac{f^3}{f^2} = \frac{p}{q}$ is a polynomial.

Problem 3. In the linear space of all real $n \times n$ matrices, find the maximum possible dimension of a linear subspace V such that

$$\forall X, Y \in V \quad \operatorname{trace}(XY) = 0.$$

(The trace of a matrix is the sum of the diagonal entries.)

Solution. If A is a nonzero symmetric matrix, then $\operatorname{trace}(A^2) = \operatorname{trace}(A^t A)$ is the sum of the squared entries of A which is positive. So V cannot contain any symmetric matrix but 0.

Denote by S the linear space of all real $n \times n$ symmetric matrices; $\dim V = \frac{n(n+1)}{2}$. Since $V \cap S = \{0\}$, we have $\dim V + \dim S \le n^2$ and thus $\dim V \le n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

The space of strictly upper triangular matrices has dimension $\frac{n(n-1)}{2}$ and satisfies the condition of the problem.

Therefore the maximum dimension of V is $\frac{n(n-1)}{2}$.

Problem 4. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is three times differentiable, then there exists a real number $\xi \in (-1, 1)$ such that

$$\frac{f'''(\xi)}{6} = \frac{f(1) - f(-1)}{2} - f'(0).$$

Solution 1. Let

$$g(x) = -\frac{f(-1)}{2} x^2(x-1) - f(0)(x^2-1) + \frac{f(1)}{2} x^2(x+1) - f'(0)x(x-1)(x+1).$$

It is easy to check that $g(\pm 1) = f(\pm 1)$, $g(0) = f(0)$ and $g'(0) = f'(0)$.

Apply Rolle's theorem for the function $h(x) = f(x) - g(x)$ and its derivatives. Since $h(-1) = h(0) = h(1) = 0$, there exist $\eta \in (-1, 0)$ and $\vartheta \in (0, 1)$ such that $h'(\eta) = h'(\vartheta) = 0$. We also have $h'(0) = 0$, so there exist $\varrho \in (\eta, 0)$ and $\sigma \in (0, \vartheta)$ such that $h''(\varrho) = h''(\sigma) = 0$. Finally, there exists a $\xi \in (\varrho, \sigma) \subset (-1, 1)$ where $h'''(\xi) = 0$. Then

$$f'''(\xi) = g'''(\xi) = -\frac{f(-1)}{2} \cdot 6 - f(0) \cdot 0 + \frac{f(1)}{2} \cdot 6 - f'(0) \cdot 6 = \frac{f(1) - f(-1)}{2} - f'(0).$$

Solution 2. The expression $\dfrac{f(1)-f(-1)}{2} - f'(0)$ is the divided difference $f[-1,0,0,1]$ and there exists a number $\xi \in (-1,1)$ such that $f[-1,0,0,1] = \dfrac{f'''(\xi)}{3!}$.

Problem 5. Find all $r > 0$ such that whenever $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function such that $|\operatorname{grad} f(0,0)| = 1$ and $|\operatorname{grad} f(u) - \operatorname{grad} f(v)| \le |u - v|$ for all $u, v \in \mathbb{R}^2$, then the maximum of f on the disk $\{u \in \mathbb{R}^2 : |u| \le r\}$ is attained at exactly one point. ($\operatorname{grad} f(u) = (\partial_1 f(u), \partial_2 f(u))$ is the gradient vector of f at the point u. For a vector $u = (a,b)$, $|u| = \sqrt{a^2+b^2}$.)

Solution. To get an upper bound for r, set $f(x,y) = x - \dfrac{x^2}{2} + \dfrac{y^2}{2}$. This function satisfies the conditions, since $\operatorname{grad} f(x,y) = (1-x, y)$, $\operatorname{grad} f(0,0) = (1,0)$ and $|\operatorname{grad} f(x_1,y_1) - \operatorname{grad} f(x_2,y_2)| = |(x_2 - x_1, y_1 - y_2)| = |(x_1,y_1) - (x_2,y_2)|$.

In the disk $D_r = \{(x,y) : x^2 + y^2 \le r^2\}$

$$f(x,y) = \frac{x^2+y^2}{2} - \left(x - \frac{1}{2}\right)^2 + \frac{1}{4} \le \frac{r^2}{2} + \frac{1}{4}.$$

If $r > \frac{1}{2}$ then the absolute maximum is $\frac{r^2}{2} + \frac{1}{4}$, attained at the points $\left(\frac{1}{2}, \pm\sqrt{r^2 - \frac{1}{4}}\right)$. Therefore, it is necessary that $r \le \frac{1}{2}$ because if $r > \frac{1}{2}$ then the maximum is attained twice.

Suppose now that $r \le 1/2$ and that f attains its maximum on D_r at u, v, $u \ne v$. Since $|\operatorname{grad} f(z) - \operatorname{grad} f(0)| \le r$, $|\operatorname{grad} f(z)| \ge 1 - r > 0$ for all $z \in D_r$. Hence f may attain its maximum only at the boundary of D_r, so we must have $|u| = |v| = r$ and $\operatorname{grad} f(u) = au$ and $\operatorname{grad} f(v) = bv$, where $a, b \ge 0$. Since $au = \operatorname{grad} f(u)$ and $bv = \operatorname{grad} f(v)$ belong to the disk D with centre $\operatorname{grad} f(0)$ and radius r, they do not belong to the interior of D_r. Hence $|\operatorname{grad} f(u) - \operatorname{grad} f(v)| = |au - bv| \ge |u - v|$ and this inequality is strict since $D \cap D_r$ contains no more than one point. But this contradicts the assumption that $|\operatorname{grad} f(u) - \operatorname{grad} f(v)| \le |u - v|$. So all $r \le \frac{1}{2}$ satisfies the condition.

Problem 6. Prove that if p and q are rational numbers and $r = p + q\sqrt{7}$, then there exists a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ne \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ with integer entries and with $ad - bc = 1$ such that

$$\frac{ar+b}{cr+d} = r.$$

Solution. First consider the case when $q = 0$ and r is rational. Choose a positive integer t such that $r^2 t$ is an integer and set

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 + rt & -r^2 t \\ t & 1 - rt \end{pmatrix}.$$

Then

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \quad \text{and} \quad \frac{ar+b}{cr+d} = \frac{(1+rt)r - r^2 t}{tr + (1 - rt)} = r.$$

Now assume $q \neq 0$. Let the minimal polynomial of r in $\mathbb{Z}[x]$ be ux^2+vx+w. The other root of this polynomial is $\overline{r} = p-q\sqrt{7}$, so $v = -u(r+\overline{r}) = -2up$ and $w = ur\overline{r} = u(p^2-7q^2)$. The discriminant is $v^2 - 4uw = 7 \cdot (2uq)^2$. The left-hand side is an integer, implying that also $\Delta = 2uq$ is an integer.

The equation $\frac{ar+b}{cr+d} = r$ is equivalent to $cr^2 + (d-a)r - b = 0$. This must be a multiple of the minimal polynomial, so we need

$$c = ut, \quad d - a = vt, \quad -b = wt$$

for some integer $t \neq 0$. Putting together these equalities with $ad - bc = 1$ we obtain that

$$(a+d)^2 = (a-d)^2 + 4ad = 4 + (v^2 - 4uw)t^2 = 4 + 7\Delta^2 t^2.$$

Therefore $4 + 7\Delta^2 t^2$ must be a perfect square. Introducing $s = a + d$, we need an integer solution (s, t) for the Diophantine equation

$$s^2 - 7\Delta^2 t^2 = 4 \tag{1}$$

such that $t \neq 0$.

The numbers s and t will be even. Then $a + d = s$ and $d - a = vt$ will be even as well and a and d will be really integers.

Let $(8 \pm 3\sqrt{7})^n = k_n \pm l_n\sqrt{7}$ for each integer n. Then $k_n^2 - 7l_n^2 = (k_n + l_n\sqrt{7})(k_n - l_n\sqrt{7}) = ((8+3\sqrt{7})^n(8-3\sqrt{7}))^n = 1$ and the sequence (l_n) also satisfies the linear recurrence $l_{n+1} = 16l_n - l_{n-1}$. Consider the residue of l_n modulo Δ. There are Δ^2 possible residue pairs for (l_n, l_{n+1}) so some are the same. Starting from such two positions, the recurrence shows that the sequence of residues is periodic in both directions. Then there are infinitely many indices such that $l_n \equiv l_0 = 0 \pmod{\Delta}$.

Taking such an index n, we can set $s = 2k_n$ and $t = 2l_n/\Delta$.

Remarks. 1. It is well-known that if $D > 0$ is not a perfect square then the Pell-like Diophantine equation

$$x^2 - Dy^2 = 1$$

has infinitely many solutions. Using this fact the solution can be generalized to all quadratic algebraic numbers.

2. It is also known that the continued fraction of a real number r is periodic from a certain point if and only if r is a root of a quadratic equation. This fact can lead to another solution.