IMC 2019, Blagoevgrad, Bulgaria

Day 1, July 30, 2019

Problem 1. Evaluate the product

$$\prod_{n=3}^{\infty} \frac{(n^3+3n)^2}{n^6-64}.$$

(10 points)

Problem 2. A four-digit number $YEAR$ is called *very good* if the system

$$\begin{aligned} Yx+Ey+Az+Rw &= Y \\ Rx+Yy+Ez+Aw &= E \\ Ax+Ry+Yz+Ew &= A \\ Ex+Ay+Rz+Yw &= R \end{aligned}$$

of linear equations in the variables x, y, z and w has at least two solutions. Find all very good YEARs in the 21st century.

(The 21st century starts in 2001 and ends in 2100.)

(10 points)

Problem 3. Let $f : (-1,1) \to \mathbb{R}$ be a twice differentiable function such that

$$2f'(x) + xf''(x) \geq 1 \quad \text{for } x \in (-1,1).$$

Prove that

$$\int_{-1}^{1} xf(x)\,\mathrm{d}x \geq \frac{1}{3}.$$

(10 points)

Problem 4. Define the sequence $a_0, a_1, \ldots$ of numbers by the following recurrence:

$$a_0 = 1, \quad a_1 = 2, \quad (n+3)a_{n+2} = (6n+9)a_{n+1} - na_n \quad \text{for } n \geq 0.$$

Prove that all terms of this sequence are integers.

(10 points)

Problem 5. Determine whether there exist an odd positive integer n and $n \times n$ matrices A and B with integer entries, that satisfy the following conditions:

1. $\det(B) = 1$;
2. $AB = BA$;
3. $A^4 + 4A^2B^2 + 16B^4 = 2019I$.

(Here I denotes the $n \times n$ identity matrix.)

(10 points)