IMC 2019, Blagoevgrad, Bulgaria

Day 2, July 31, 2019

Problem 6. Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous functions such that g is differentiable. Assume that $\big(f(0) - g'(0)\big)\big(g'(1) - f(1)\big) > 0$. Show that there exists a point $c \in (0,1)$ such that $f(c) = g'(c)$.

Proposed by Fereshteh Malek, K. N. Toosi University of Technology

Solution. Define $F(x) = \int_0^x f(t)\mathrm{d}t$ and let $h(x) = F(x) - g(x)$. By the continuouity of f we have $F' = f$, so $h' = f - g'$.

The assumption can be re-written as $h'(0)\big(-h'(1)\big) > 0$, so $h'(0)$ and $h'(1)$ have opposite signs. Then, by the Mean Value Theorem For Derivatives (Darboux property of derivatives) it follows that there is a point c between 0 and 1 where $h'(c) = 0$, so $f(c) = g'(c)$.

Problem 7.

Let $C = \{4, 6, 8, 9, 10, \ldots\}$ be the set of composite positive integers. For each $n \in C$ let a_n be the smallest positive integer k such that $k!$ is divisible by n. Determine whether the following series converges:

$$\sum_{n \in C} \left(\frac{a_n}{n}\right)^n. \tag{1}$$

Proposed by Orif Ibrogimov, ETH Zurich and National University of Uzbekistan

Solution. The series (1) converges. We will show that $\frac{a_n}{n} \le \frac{2}{3}$ for $n > 4$; then the geometric series $\sum \left(\frac{2}{3}\right)^n$ majorizes (1).

Case 1: n has at least two distinct prime divisors. Then n can be factored as $n = qr$ with some co-prime positive integers $q, r \ge 2$; without loss of generality we can assume $q > r$. Notice that $q \,\big|\, q!$ and $r \,\big|\, r! \,\big|\, q!$, so $n = qr \,\big|\, q!$; this shows $a_n \le q$ and therefore

$$\frac{a_n}{n} \le \frac{q}{n} = \frac{1}{r} \le \frac{1}{2}.$$

Case 2: n is the square of a prime, $n = p^2$ with some prime $p \ge 3$. From $p^2 \,\big|\, p \cdot 2p \,\big|\, (2p)!$ we obtain $a_n = 2p$, so

$$\frac{a_n}{n} = \frac{2p}{p^2} = \frac{2}{p} \le \frac{2}{3}.$$

Case 3: n is a prime power, $n = p^k$ with some prime p and $k \ge 3$. Notice that $n = p^k \,\big|\, p \cdot p^2 \cdots p^{k-1}$, so $a_n \le p^{k-1}$ and therefore

$$\frac{a_n}{n} \le \frac{p^{k-1}}{p^k} = \frac{1}{p} \le \frac{1}{2}.$$

Problem 8. Let $x_1, \dots, x_n$ be real numbers. For any set $I \subset \{1, 2, \dots, n\}$ let $s(I) = \sum\limits_{i \in I} x_i$. Assume that the function $I \mapsto s(I)$ takes on at least 1.8^n values where I runs over all 2^n subsets of $\{1, 2, \dots, n\}$. Prove that the number of sets $I \subset \{1, 2, \dots, n\}$ for which $s(I) = 2019$ does not exceed 1.7^n.

Proposed by Fedor Part and Fedor Petrov, St. Petersburg State University

Solution. Choose disctint sets $I_1, \dots, I_A \subset \{1, 2, \dots, n\}$ where $A \geq 1.8^n$, and let $J_1, \dots, J_B \subset \{1, 2, \dots, n\}$ be all sets so that $S(J_i) = 2019$; for the sake of contradiction, assume that $B \geq 1.7^n$.

Every set $I \subset \{1, 2, \dots, n\}$ can be identified with a $0 - 1$ vector of length n: the kth coordinate in the vector is 1 if $k \in I$. Then $s(I) = \langle I, X \rangle$, where $X = (x_1, \dots, x_n)$ and $\langle \cdot, \cdot \rangle$ stands for the usual scalar product.

For all ordered pairs $(a, b) \in \{1, \dots, A\} \times \{1, \dots, B\}$ consider the vector $I_a - J_b \in \{-1, 0, 1\}^n$. By the pigeonhole principle, since $AB \geq (1.8 \cdot 1.7)^n > 3^n$, there are two pairs (a, b) and (c, d) such that $I_a - J_b = I_c - J_d$. Multiplying this by X we get $s(I_a) - 2019 = s(I_c) - 2019$; that implies $a = c$. But then $J_b = J_d$, that is, $b = d$, and our pairs coincide. Contradiction.

Problem 9. Determine all positive integers n for which there exist $n \times n$ real invertible matrices A and B that satisfy $AB - BA = B^2 A$.

Proposed by Karen Keryan, Yerevan State University & American University of Armenia, Yerevan

Solution. We prove that there exist such matrices A and B inf and only if n is even.

I. Assume that n is odd and some invertible $n \times n$ matrices A, B satisfy $AB - BA = B^2 A$. Hence $B = A^{-1}(B^2 + B)A$, so the matrices B and $B^2 + B$ are similar and therefore have the same eigenvalues. Since n is odd, the matrix B has a real eigenvalue, denote it by λ_1. Therefore $\lambda_2 := \lambda_1^2 + \lambda_1$ is an eigenvalue of $B^2 + B$, hence an eigenvalue of B. Similarly, $\lambda_3 := \lambda_2^2 + \lambda_2$ is an eigenvalue of $B^2 + B$, hence an eigenvalue of B. Repeating this process and taking into account that the number of eigenvalues of B is finite we will get there exist numbers $k \leq l$ so that $\lambda_{l+1} = \lambda_k$. Hence

$$\begin{aligned}
\lambda_{k+1} &= \lambda_k^2 + \lambda_k \\
\lambda_{k+2} &= \lambda_{k+1}^2 + \lambda_{k+1} \\
&\dots\dots\dots \\
\lambda_l &= \lambda_{l-1}^2 + \lambda_{l-1} \\
\lambda_k &= \lambda_l^2 + \lambda_l.
\end{aligned}$$

Adding this equations we get $\lambda_k^2 + \lambda_{k+1}^2 + \ldots + \lambda_l^2 = 0$. Taking into account that all λ_i's are real (as λ_1 is real), we have $\lambda_k = \ldots = \lambda_l = 0$, which implies that B is not invertible, contradiction.

II. Now we construct such matrices A, B for even n. Let $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B_2 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$. It is easy to check that the matrices A_2, B_2 are invertible and satisfy the condition. For $n = 2k$ the $n \times n$ block matrices

$$A = \begin{bmatrix} A_2 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_2 & 0 & \dots & 0 \\ 0 & B_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_2 \end{bmatrix}$$

are also invertible and satisfy the condition.

Problem 10. 2019 points are chosen at random, independently, and distributed uniformly in the unit disc $\{(x,y) \in \mathbb{R}^2 \colon x^2+y^2 \le 1\}$. Let C be the convex hull of the chosen points. Which probability is larger: that C is a polygon with three vertices, or a polygon with four vertices?

Proposed by Fedor Petrov, St. Petersburg State University

Solution. We will show that the quadrilateral has larger probability.

Let $\mathcal{D} = \{(x,y) \in \mathbb{R}^2 \colon x^2+y^2 \le 1\}$. Denote the random points by $X_1, \dots, X_{2019}$ and let

$$p = P\big(C \text{ is a triangle with vertices } X_1, X_2, X_3\big),$$
$$q = P\big(C \text{ is a convex quadrilateral with vertices } X_1, X_2, X_3, X_4\big).$$

By symmetry we have $P\big(C \text{is a triangle}\big) = \binom{2019}{3}p$, $P\big(C \text{is a quadrilateral}\big) = \binom{2019}{4}q$ and we need to prove that $\binom{2019}{4}q > \binom{2019}{3}p$, or equivalently $p < \frac{2016}{4}q = 504q$.

Note that p is the average over X_1, X_2, X_3 of the following expression:

$$u(X_1,X_2,X_3) = P\big(X_4 \in \triangle X_1X_2X_3\big) \cdot P\big(X_5, X_6, \dots, X_{2019} \in \triangle X_1X_2X_3\big),$$

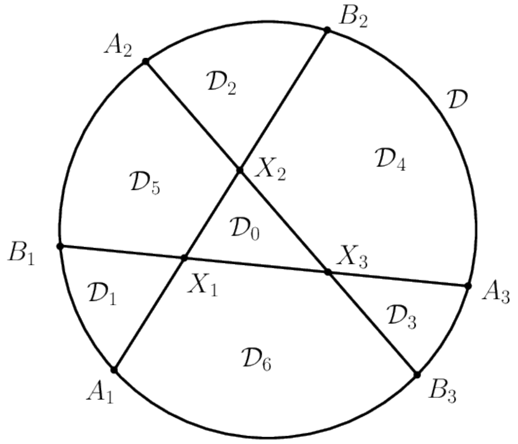
and q is not less than the average over X_1, X_2, X_3 of

$$v(X_1,X_2,X_3) = P\big(X_1,X_2,X_3,X_4 \text{ form a convex quad.}\big) \cdot P\big(X_5, X_6, \dots, X_{2019} \in \triangle X_1X_2X_3)\big).$$

Thus it suffices to prove that $u(X_1,X_2,X_3) \le 500v(X_1,X_2,X_3)$ for all X_1, X_2, X_3. It reads as $\text{area}(\triangle X_1X_2X_3) \le 500\text{area}(\Omega)$, where $\Omega = \{Y : X_1, X_2, X_3, Y \text{ form a convex quadrilateral}\}$.

Assume the contrary, i.e., $\text{area}(\triangle X_1X_2X_3) > 500\text{area}(\Omega)$.

Let the lines X_1X_2, X_1X_3, X_2X_3 meet the boundary of $\mathcal{D}$ at $A_1, A_2, A_3, B_1, B_2, B_3$; these lines divide $\mathcal{D}$ into 7 regions as shown in the picture; $\Omega = \mathcal{D}_4 \cup \mathcal{D}_5 \cup \mathcal{D}_6$.



By our indirect assumption,

$$\text{area}(\mathcal{D}_4) + \text{area}(\mathcal{D}_5) + \text{area}(\mathcal{D}_6) = \text{area}(\Omega) < \frac{1}{500}\text{area}(\mathcal{D}_0) < \frac{1}{500}\text{area}(\mathcal{D}) = \frac{\pi}{500}.$$

From $\triangle X_1X_3B_3 \subset \Omega$ we get $X_3B_3/X_3X_2 = \text{area}(\triangle X_1X_3B_3)/\text{area}(\triangle X_1X_2X_3) < 1/500$, so $X_3B_3 < \frac{1}{500}X_2X_3 < \frac{1}{250}$. Similarly, the lengths segments $A_1X_1, B_1X_1, A_2X_2, B_2X_2, A_3X_2$ are less than $\frac{1}{250}$.

The regions $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ can be covered by disks with radius $\frac{1}{250}$, so

$$\text{area}(\mathcal{D}_1) + \text{area}(\mathcal{D}_2) + \text{area}(\mathcal{D}_3) < 3 \cdot \frac{\pi}{250^2}.$$

Finally, it is well-known that the area of any triangle inside the unit disk is at most $\frac{3\sqrt{3}}{4}$, so

$$\text{area}(\mathcal{D}_0) \le \frac{3\sqrt{3}}{4}.$$

But then

$$\sum_{i=0}^{6} \operatorname{area}(\mathcal{D}_i) < \frac{3\sqrt{3}}{4} + 3 \cdot \frac{\pi}{250^2} + \frac{\pi}{500} < \operatorname{area}(\mathcal{D}),$$

contradiction.