IMC 2023

Second Day, August 3, 2023

Problem 6. Ivan writes the matrix $\begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$ on the board. Then he performs the following operation on the matrix several times:

- he chooses a row or a column of the matrix, and
- he multiplies or divides the chosen row or column entry-wise by the other row or column, respectively.

Can Ivan end up with the matrix $\begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}$ after finitely many steps?

(10 points)

Problem 7. Let V be the set of all continuous functions $f\colon [0,1] \to \mathbb{R}$, differentiable on $(0,1)$, with the property that $f(0) = 0$ and $f(1) = 1$. Determine all $\alpha \in \mathbb{R}$ such that for every $f \in V$, there exists some $\xi \in (0,1)$ such that

$$f(\xi) + \alpha = f'(\xi).$$

(10 points)

Problem 8. Let T be a tree with n vertices; that is, a connected simple graph on n vertices that contains no cycle. For every pair u, v of vertices, let $d(u,v)$ denote the distance between u and v, that is, the number of edges in the shortest path in T that connects u with v.

Consider the sums

$$W(T) = \sum_{\substack{\{u,v\} \subseteq V(T) \\ u \neq v}} d(u,v) \quad \text{and} \quad H(T) = \sum_{\substack{\{u,v\} \subseteq V(T) \\ u \neq v}} \frac{1}{d(u,v)}.$$

Prove that

$$W(T) \cdot H(T) \geq \frac{(n-1)^3(n+2)}{4}.$$

(10 points)

Problem 9. We say that a real number V is *good* if there exist two closed convex subsets X, Y of the unit cube in $\mathbb{R}^3$, with volume V each, such that for each of the three coordinate planes (that is, the planes spanned by any two of the three coordinate axes), the projections of X and Y onto that plane are disjoint.

Find $\sup\{V \mid V \text{ is good}\}$.

(10 points)

Problem 10. For every positive integer n, let $f(n), g(n)$ be the minimal positive integers such that

$$1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} = \frac{f(n)}{g(n)}.$$

Determine whether there exists a positive integer n for which $g(n) > n^{0.999\,n}$.

(10 points)