IMC 2023

Second Day, August 3, 2023

Solutions

Problem 6. Ivan writes the matrix $\begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$ on the board. Then he performs the following operation on the matrix several times:

- he chooses a row or a column of the matrix, and
- he multiplies or divides the chosen row or column entry-wise by the other row or column, respectively.

Can Ivan end up with the matrix $\begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}$ after finitely many steps?

(proposed by Alex Avdiushenko, Neapolis University Paphos, Cyprus)

Hint: Construct an invariant quantity that does not change during Ivan's prcedure.

Solution. We show that starting from $A = \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$, Ivan cannot reach the matrix $B = \begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}$.

Notice first that the allowed operations preserve the positivity of entries; all matrices Ivan can reach have only positive entries.

For every matrix $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ with positive entries, let $L(X) = \begin{pmatrix} \log_2 x_{11} & \log_2 x_{12} \\ \log_2 x_{21} & \log_2 x_{22} \end{pmatrix}$. By taking logarithms of the entries, the steps in Ivan game will be replaced by adding or subtracting a row or column to the other row. Such standard row and column operations preserve the determinant.

Hence, if the matrices in the game are $A = X_0, X_1, X_2, \ldots$, then we have $\det L(A) = \det L(X_1) = \det L(X_2) = \ldots$, and it suffices to verify that $\det L(A) \neq \det L(B)$.

Indeed,

$$\det L(A) = \log_2 2 \cdot \log_2 4 - \log_2 4 \cdot \log_2 3 = \log_2(4/3) > 0$$

and similarly $\det L(B) < 0$, so $\det L(A) \neq \det L(B)$.

Problem 7. Let V be the set of all continuous functions $f\colon [0,1] \to \mathbb{R}$, differentiable on $(0,1)$, with the property that $f(0) = 0$ and $f(1) = 1$. Determine all $\alpha \in \mathbb{R}$ such that for every $f \in V$, there exists some $\xi \in (0,1)$ such that

$$f(\xi) + \alpha = f'(\xi).$$

(proposed by Mike Daas, Leiden University)

Hint: Find a function $h \in V$ such that $h' - h$ is constant, then apply Rolle's theorem to $f - h$.
Alternatively, you can apply Cauchys's mean value theorem with some auxiliary functions.

Solution 1. First consider the function

$$h(x) = \frac{e^x - 1}{e - 1}, \quad \text{which has the property that} \quad h'(x) = \frac{e^x}{e-1}.$$

Note that $h \in V$ and that $h'(x) - h(x) = 1/(e-1)$ is constant. As such, $\alpha = 1/(e-1)$ is the only possible value that could possibly satisfy the condition from the problem. For $f \in V$ arbitrary, let

$$g(x) = f(x)e^{-x} + h(-x), \quad \text{with} \quad g(0) = 0 \quad \text{and also} \quad g(1) = e^{-1} + \frac{e^{-1} - 1}{e - 1} = 0.$$

We compute that

$$g'(x) = f'(x)e^{-x} - f(x)e^{-x} - h'(-x).$$

Now apply Rolle's Theorem to g on the interval $[0,1]$; it yields some $\xi \in (0,1)$ with the property that

$$g'(\xi) = 0 \implies f'(\xi)e^{-\xi} - f(\xi)e^{-\xi} - \frac{e^{-\xi}}{e-1} = 0 \implies f'(\xi) = f(\xi) + \frac{1}{e-1},$$

showing that $\alpha = 1/(e-1)$ indeed satisfies the condition from the problem.

Solution 2. Notice that the expression $f'(x) - f(x)$ appears in the derivative of the function $F(x) = f(x) \cdot e^{-x}$: $F'(x) = \big(f'(x) - f(x)\big)e^{-x}$.

Apply Cauchy's mean value theorem to $F(x)$ and the function $G(x) = -e^{-x}$. By the theorem, there is some $\xi \in (0,1)$ such that

$$\frac{F'(\xi)}{G'(\xi)} = \frac{F(1) - F(0)}{G(1) - G(0)}$$

$$f'(\xi) - f(\xi) = \frac{e^{-1} - 0}{-e^{-1} + 1} = \frac{1}{e-1}.$$

This proves the required property for $a = \dfrac{1}{e-1}$.

Now we show that no other α is possible. Choose f and F in such a way that $\dfrac{F'(x)}{G'(x)} = f'(x) - f(x) = \dfrac{1}{e-1}$ is constant. That means

$$F'(x) = \frac{G'(x)}{e-1} = \frac{e^{-x}}{e-1},$$

$$F(x) = \frac{1 - e^{-x}}{e-1},$$

$$f(x) = F(x) \cdot e^x = \frac{e^x - 1}{e-1}.$$

With this choice we have $f(0) = 0$ and $f(1) = 1$, so $f' \in V$, and $f'(x) - f(x) \equiv \dfrac{1}{e-1}$ for all x, so for this function the only possible value for α is $\dfrac{1}{e-1}$.

Problem 8. Let T be a tree with n vertices; that is, a connected simple graph on n vertices that contains no cycle. For every pair u, v of vertices, let $d(u,v)$ denote the distance between u and v, that is, the number of edges in the shortest path in T that connects u with v.

Consider the sums

$$W(T) = \sum_{\substack{\{u,v\}\subseteq V(T)\\ u\neq v}} d(u,v) \quad \text{and} \quad H(T) = \sum_{\substack{\{u,v\}\subseteq V(T)\\ u\neq v}} \frac{1}{d(u,v)}.$$

Prove that

$$W(T)\cdot H(T) \geq \frac{(n-1)^3(n+2)}{4}.$$

(proposed by Slobodan Filipovski, University of Primorska, Koper)

Hint: There are $n-1$ pairs u, v with $d(u,v) = 1$; in all other cases $d(u,v) \geq 2$.

Solution. Let $k = \binom{n}{2}$ and let $x_1 \leq x_2 \leq \ldots \leq x_k$ be the distances between the pairs of vertices in the tree T. Thus

$$W(T)\cdot H(T) = (x_1 + x_2 + \ldots + x_k)\cdot\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_k}\right).$$

Since the tree has exactly $n-1$ edges, there are exactly $n-1$ pairs of vertices at distance one, that is, $x_1 = x_2 = \ldots = x_{n-1} = 1$. Thus

$$W(T)\cdot H(T) = (n-1+x_n+x_{n+1}+\ldots+x_k)\cdot\left(n-1+\frac{1}{x_n}+\frac{1}{x_{n+1}}+\ldots+\frac{1}{x_k}\right) =$$

$$= (n-1)^2 + (n-1)\left(\left(x_n+\frac{1}{x_n}\right)+\ldots+\left(x_k+\frac{1}{x_k}\right)\right)+$$

$$+(x_n+\ldots+x_k)\left(\frac{1}{x_n}+\ldots+\frac{1}{x_k}\right).$$

From Cauchy inequality we have

$$(x_n+\ldots+x_k)\left(\frac{1}{x_n}+\ldots+\frac{1}{x_k}\right) \geq (1+1+\ldots+1)^2 = (k-n+1)^2 = \frac{(n-1)^2(n-2)^2}{4}.$$

The equality holds if and only if $x_n = x_{n+1} = \ldots = x_k$.
Now we minimize the expression $\left(x_n+\frac{1}{x_n}\right)+\ldots+\left(x_k+\frac{1}{x_k}\right)$, where $x_i \in [2, n-1]$.
It is clear that the minimal value is achieved for $x_n = x_{n+1} = \ldots = x_k = 2$. Therefore we get

$$W(T)\cdot H(T) \geq (n-1)^2 + (n-1)\left(\left(2+\frac{1}{2}\right)(k-n+1)\right) + \frac{(n-1)^2(n-2)^2}{4} = \frac{(n-1)^3(n+2)}{4}.$$

The equality holds for $x_1 = \ldots = x_{n-1} = 1$ and $x_n = x_{n+1} = \ldots = x_k = 2$, that is, the smallest value is achieved for the tree where $n-1$ pairs are at distance one, and the remaining $k-(n-1) = \frac{(n-1)(n-2)}{2}$ pairs are at distance two. The unique tree which satisfies these conditions is the star graph S_n. In this case it holds

$$W(S_n)\cdot H(S_n) = (n-1)^2\cdot\frac{(n-1)(n+2)}{4} = \frac{(n-1)^3(n+2)}{4}.$$

Problem 9. We say that a real number V is *good* if there exist two closed convex subsets X, Y of the unit cube in $\mathbb{R}^3$, with volume V each, such that for each of the three coordinate planes (that is, the planes spanned by any two of the three coordinate axes), the projections of X and Y onto that plane are disjoint.

Find $\sup\{V \mid V \text{ is good}\}$.

(proposed by Josef Tkadlec and Arseniy Akopyan)

Hint: The two bodies can be replaced by a pair symmetric to the midpoint of the cube.

Solution. We prove that $\sup\{V \mid V \text{ is good}\} = 1/4$.

We will use the unit cube $U = [-1/2, 1/2]^3$.

For $\varepsilon \to 0$, the axis-parallel boxes $X = [-1/2, -\varepsilon] \times [-1/2, -\varepsilon] \times [-1/2, 1/2]$ and $Y = [\varepsilon, 1/2] \times [\varepsilon, 1/2] \times [-1/2, 1/2]$ show that $\sup\{V\} \geq 1/4$.

To prove the other bound, consider two admissible convex bodies X, Y. For any point $P = [x, y, z] \in U$ with $xyz \neq 0$, let $\overline{P} = \{[\pm x, \pm y, \pm z]\}$ be the set consisting of 8 points (the original P and its 7 "symmetric" points). If for each such P we have $|\overline{P} \cap (X \cup Y)| \leq 4$, then the conclusion follows by integrating. Suppose otherwise and let P be a point with $|\overline{P} \cap (X \cup Y)| \geq 5$. Below we will complete the proof by arguing that:

(1) we can replace one of the two bodies (the "thick" one) with the reflection of the other body about the origin, and

(2) for such symmetric pairs of bodies we in fact have $|\overline{P} \cap (X \cup Y)| \leq 4$, for all P.

To prove Claim (1), we say that a convex body is *thick* if each of its three projections contains the origin. We claim that one of the two bodies X, Y is thick. This is a short casework on the 8 points of $\overline{P}$. Since $|\overline{P} \cap (X \cup Y)| \geq 5$, by pigeonhole principle, we find a pair of points in $\overline{P} \cap (X \cup Y)$ symmetric about the origin. If both points belong to one body (say to X), then by convexity of X the origin belongs to X, thus X is thick. Otherwise, label $\overline{P}$ as $ABCDA'B'C'D'$. Wlog $A \in X$, $C' \in Y$ is the pair of points in $\overline{P}$ symmetric about the origin. Wlog at least 3 points of $\overline{P}$ belong to X. Since X, Y have disjoint projections, we have $C, B', D' \notin X$, so wlog $B, D \in X$. Then Y can contain no other point of $\overline{P}$ (apart from C'), so X must contain at least 4 points of $\overline{P}$ and thus $A' \in X$. But then each projection of X contains the origin, so X is indeed thick.

Note that if X is thick then none of the three projections of Y contains the origin. Consider the reflection $Y' = -Y$ of Y about the origin. Then (Y, Y') is an admissible pair with the same volume as (X, Y): the two bodies Y and Y' clearly have equal volumes V and they have disjoint projections (by convexity, since the projections of Y miss the origin). This proves Claim (1).

Claim (2) follows from a similar small casework on the 8-tuple $\overline{P}$: For contradiction, suppose $|\overline{P} \cap Y'| = |\overline{P} \cap Y| \geq 3$. Wlog $A \in Y'$. Then $C' \in Y$, so $C, B', D' \notin Y'$, so wlog $B, D \in Y'$. Then $B', D' \in Y$, a contradiction with (Y, Y') being admissible.

Remark. There are more examples with $V \to 1/4$, e.g. X a union of two triangular pyramids with base ACD' – one with apex D, one with apex at the origin (and Y symmetric with X about the origin).

Remark. The word "convex" matters. E.g., in a $3 \times 3 \times 3$ cube, one can set X to be a $2 \times 2 \times 2$ sub-cube, and Y to be the (non-convex) 3D L-shape consisting of 7 unit cubes. This shows that without convexity we have $V \geq 7/27 > 1/4$.

Problem 10. For every positive integer n, let $f(n), g(n)$ be the minimal positive integers such that

$$1+\frac{1}{1!}+\frac{1}{2!}+\ldots+\frac{1}{n!}=\frac{f(n)}{g(n)}.$$

Determine whether there exists a positive integer n for which $g(n)>n^{0.999\,n}$.

(proposed by Fedor Petrov, St. Petersburg State University)

Solution. We show that there does exist such a number n.

Let $\varepsilon=10^{-10}$. Call a prime p *special*, if for certain $k\in\{1,2,\ldots,p-1\}$ there exist at least $\varepsilon\cdot k$ positive integers $j\le k$ for which p divides $f(j)$.

Lemma. There exist only finitely many special primes.

Proof. Let p be a special prime number, and p divides $f(j)$ for at least $\varepsilon\cdot k$ values of $j\in\{1,2,\ldots,k\}$. Note that if p divides $f(j)$ and $f(j+r)$, then p divides

$$(j+r)!\left(\frac{f(j+r)}{g(j+r)}-\frac{f(j)}{g(j)}\right)=1+(j+r)+(j+r)(j+r-1)+\ldots+(j+r)\ldots(j+2)$$

that is a polynomial of degree $r-1$ with respect to j. Thus, for fixed j it equals to 0 modulo p for at most $r-1$ values of j. Look at our $\ge\varepsilon\cdot k$ values of $j\in\{1,2,\ldots,k\}$ and consider the gaps between consecutive j's. The number of such gaps which are greater than $2/\varepsilon$ does not exceed $\varepsilon\cdot k/2$ (since the total sum of gaps is less than k). Therefore, at least $\varepsilon\cdot k/2-1$ gaps are at most $2/\varepsilon$. But the number of such small gaps is bounded from above by a constant (not depending on k) by the above observation. Therefore, k is bounded, and, since p divides $f(1)f(2)\ldots f(k)$, p is bounded too.

Now we want to bound the product $g(1)g(2)\ldots g(n)$ (for a large integer n) from below. Let $p\le n$ be a non-special prime. Our nearest goal is to prove that

$$\nu_p\big(g(1)g(2)\ldots g(n)\big)\ge(1-\varepsilon)\nu_p(1!\cdot 2!\cdot\ldots\cdot n!) \tag{1}$$

Partition the numbers $p,p+1,\ldots,n$ onto the intervals of length p (except possibly the last interval which may be shorter): $\{p,p+1,\ldots,2p-1\}$, $\ldots$, $\{p\lfloor n/p\rfloor,\ldots,n\}$. Note that in every interval $\Delta=[a\cdot p,a\cdot p+k]$, all factorials $x!$ with $x\in\Delta$ have the same p-adic valuation, denote it $T=\nu_p\big((ap)!\big)$. We claim that at least $(1-\varepsilon)(k+1)$ valuations of $g(x)$, $x\in\Delta$, are equal to the same number T. Indeed, if $j=0$ or $1\le j\le k$ and $f(j)$ is not divisible by p, then

$$\frac{1}{(ap)!}+\frac{1}{(ap+1)!}+\ldots+\frac{1}{(ap+j)!}=\frac{1}{(ap)!}\cdot\frac{A}{B}$$

where $A\equiv f(j)\pmod p$, $B\equiv g(j)\pmod p$, so, this sum has the same p-adic valuation as $1/(ap)!$, which is strictly less than that of the sum $\sum_{i=0}^{ap-1}1/i!$, that yields $\nu_p\big(g(ap+j)\big)=\nu_p\big((ap)!\big)$. Using this for every segment Δ, we get (1).

Now, using (1) for all non-special primes, we get

$$A\cdot g(1)g(2)\ldots g(n)\ge(1!\cdot 2!\cdot\ldots\cdot n!)^{1-\varepsilon},$$

where $A=\prod_{p,k}p^{\nu_p(g(k))}$, p runs over non-special primes, k from 1 to n. Since $\nu_p(g(k))\le\nu_p(k!)=\sum_{i=1}^{\infty}\lfloor k/p^i\rfloor\le k$, we get

$$A\le\Big(\prod_p p\Big)^{1+2+\ldots+n}\le C^{n^2}$$

for some constant C. But if we had $g(n)\le n^{0.999n}\le e^n n!^{0.999}$ for all n, then

$$\log\big(A\cdot g(1)g(2)\ldots g(n)\big)\le O(n^2)+0.999\log(1!\cdot 2!\cdot\ldots\cdot n!)<(1-\varepsilon)\log(1!\cdot 2!\cdot\ldots\cdot n!)$$

for large n, a contradiction.