IMC2019: Problems on Day 1

Problem 1. Evaluate the product

$$\prod_{n=3}^{\infty} \frac{(n^3+3n)^2}{n^6-64}.$$

Proposed by Orif Ibrogimov, ETH Zurich and National University of Uzbekistan a

Karen Keryan, Yerevan State University and American University of Armenia, Yere

Hint Solution

Problem 2. A four-digit number YEAR is called *very good* if the system

$$\begin{aligned} Yx + Ey + Az + Rw &= Y \\ Rx + Yy + Ez + Aw &= E \\ Ax + Ry + Yz + Ew &= A \\ Ex + Ay + Rz + Yw &= R \end{aligned}$$

of linear equations in the variables x, y, z and w has at least two solutions. Find all very good YEARs in the 21st century.

(The 21st century starts in 2001 and ends in 2100.)

Proposed by Tomáš Bárta, Charles University, Prag

Hint Solution

Problem 3. Let $f : (-1,1) \to \mathbb{R}$ be a twice differentiable function such that

$$2f'(x) + xf''(x) \geq 1 \quad \text{for } x \in (-1,1).$$

Prove that

$$\int_{-1}^{1} xf(x)\,dx \geq \frac{1}{3}.$$

Proposed by Orif Ibrogimov, ETH Zurich and National University of Uzbekistan a

Karim Rakhimov, Scuola Normale Superiore and National University of Uzbekis

Hint Solution

Problem 4. Define the sequence $a_0, a_1, \ldots$ of numbers by the following recurrence:

$$a_0 = 1, \quad a_1 = 2, \quad (n+3)a_{n+2} = (6n+9)a_{n+1} - na_n \quad \text{for } n \geq 0.$$

Prove that all terms of this sequence are integers.

Proposed by Khakimboy Egamberganov, ICTP, It

Hint Solution

Problem 5. Determine whether there exist an odd positive integer n and $n \times n$ matrices A and B with integer entries, that satisfy the following conditions:

(1) $\det(B) = 1$;

(2) $AB = BA$;

(3) $A^4 + 4A^2B^2 + 16B^4 = 2019I$.

(Here I denotes the $n \times n$ identity matrix.)

Proposed by Orif Ibrogimov, ETH Zurich and National University of Uzbekis

Hint Solution

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