International Competition in Mathematics for Universtiy Students in Plovdiv, Bulgaria 1995

PROBLEMS AND SOLUTIONS

First day

Problem 1. (10 points)

Let X be a nonsingular matrix with columns $X_1, X_2, \dots, X_n$. Let Y be a matrix with columns $X_2, X_3, \dots, X_n, 0$. Show that the matrices $A = YX^{-1}$ and $B = X^{-1}Y$ have rank $n-1$ and have only 0's for eigenvalues.

Solution. Let $J = (a_{ij})$ be the $n \times n$ matrix where $a_{ij} = 1$ if $i = j+1$ and $a_{ij} = 0$ otherwise. The rank of J is $n-1$ and its only eigenvalues are $0'$s. Moreover $Y = XJ$ and $A = YX^{-1} = XJX^{-1}$, $B = X^{-1}Y = J$. It follows that both A and B have rank $n-1$ with only $0'$s for eigenvalues.

Problem 2. (15 points)

Let f be a continuous function on $[0,1]$ such that for every $x \in [0,1]$ we have $\displaystyle\int_x^1 f(t)dt \geq \frac{1-x^2}{2}$. Show that $\displaystyle\int_0^1 f^2(t)dt \geq \frac{1}{3}$.

Solution. From the inequality

$$0 \leq \int_0^1 (f(x) - x)^2 \, dx = \int_0^1 f^2(x)dx - 2\int_0^1 xf(x)dx + \int_0^1 x^2 dx$$

we get

$$\int_0^1 f^2(x)dx \geq 2\int_0^1 xf(x)dx - \int_0^1 x^2 dx = 2\int_0^1 xf(x)dx - \frac{1}{3}.$$

From the hypotheses we have $\displaystyle\int_0^1 \int_x^1 f(t)dtdx \geq \int_0^1 \frac{1-x^2}{2}dx$ or $\displaystyle\int_0^1 tf(t)dt \geq \frac{1}{3}$. This completes the proof.

Problem 3. (15 points)

Let f be twice continuously differentiable on $(0, +\infty)$ such that $\lim\limits_{x\to 0+} f'(x) = -\infty$ and $\lim\limits_{x\to 0+} f''(x) = +\infty$. Show that

$$\lim_{x\to 0+} \frac{f(x)}{f'(x)} = 0.$$

Solution. Since f' tends to $-\infty$ and f'' tends to $+\infty$ as x tends to $0+$, there exists an interval $(0,r)$ such that $f'(x) < 0$ and $f''(x) > 0$ for all $x \in (0,r)$. Hence f is decreasing and f' is increasing on $(0,r)$. By the mean value theorem for every $0 < x < x_0 < r$ we obtain

$$f(x) - f(x_0) = f'(\xi)(x - x_0) > 0,$$

for some $\xi \in (x, x_0)$. Taking into account that f' is increasing, $f'(x) < f'(\xi) < 0$, we get

$$x - x_0 < \frac{f'(\xi)}{f'(x)}(x - x_0) = \frac{f(x) - f(x_0)}{f'(x)} < 0.$$

Taking limits as x tends to $0+$ we obtain

$$-x_0 \leq \liminf_{x \to 0+} \frac{f(x)}{f'(x)} \leq \limsup_{x \to 0+} \frac{f(x)}{f'(x)} \leq 0.$$

Since this happens for all $x_0 \in (0,r)$ we deduce that $\lim_{x \to 0+} \frac{f(x)}{f'(x)}$ exists and $\lim_{x \to 0+} \frac{f(x)}{f'(x)} = 0$.

Problem 4. (15 points)
Let $F : (1, \infty) \to \mathbb{R}$ be the function defined by

$$F(x) := \int_x^{x^2} \frac{dt}{\ln t}.$$

Show that F is one-to-one (i.e. injective) and find the range (i.e. set of values) of F.

Solution. From the definition we have

$$F'(x) = \frac{x - 1}{\ln x}, \quad x > 1.$$

Therefore $F'(x) > 0$ for $x \in (1, \infty)$. Thus F is strictly increasing and hence one-to-one. Since

$$F(x) \geq (x^2 - x) \min \left\{ \frac{1}{\ln t} \ : \ x \leq t \leq x^2 \right\} = \frac{x^2 - x}{\ln x^2} \to \infty$$

as $x \to \infty$, it follows that the range of F is $(F(1+), \infty)$. In order to determine $F(1+)$ we substitute $t = e^v$ in the definition of F and we get

$$F(x) = \int_{\ln x}^{2\ln x} \frac{e^v}{v} dv.$$

Hence

$$F(x) < e^{2\ln x} \int_{\ln x}^{2\ln x} \frac{1}{v} dv = x^2 \ln 2$$

and similarly $F(x) > x \ln 2$. Thus $F(1+) = \ln 2$.

Problem 5. (20 points)

Let A and B be real $n \times n$ matrices. Assume that there exist $n+1$ different real numbers $t_1, t_2, \dots, t_{n+1}$ such that the matrices

$$C_i = A + t_i B, \quad i = 1, 2, \dots, n+1,$$

are nilpotent (i.e. $C_i^n = 0$).

Show that both A and B are nilpotent.

Solution. We have that

$$(A + tB)^n = A^n + tP_1 + t^2 P_2 + \cdots + t^{n-1} P_{n-1} + t^n B^n$$

for some matrices $P_1, P_2, \dots, P_{n-1}$ not depending on t.

Assume that $a, p_1, p_2, \dots, p_{n-1}, b$ are the (i,j)-th entries of the corresponding matrices $A^n, P_1, P_2, \dots, P_{n-1}, B^n$. Then the polynomial

$$bt^n + p_{n-1} t^{n-1} + \cdots + p_2 t^2 + p_1 t + a$$

has at least $n+1$ roots $t_1, t_2, \dots, t_{n+1}$. Hence all its coefficients vanish. Therefore $A^n = 0$, $B^n = 0$, $P_i = 0$; and A and B are nilpotent.

Problem 6. (25 points)

Let $p > 1$. Show that there exists a constant $K_p > 0$ such that for every $x, y \in \mathbb{R}$ satisfying $|x|^p + |y|^p = 2$, we have

$$(x-y)^2 \le K_p \left(4 - (x+y)^2\right).$$

Solution. Let $0<\delta<1$. First we show that there exists $K_{p,\delta}>0$ such that

$$f(x,y)=\frac{(x-y)^2}{4-(x+y)^2}\le K_{p,\delta}$$

for every $(x,y)\in D_\delta=\{(x,y)\ :\ |x-y|\ge\delta, |x|^p+|y|^p=2\}$.

Since D_δ is compact it is enough to show that f is continuous on D_δ. For this we show that the denominator of f is different from zero. Assume the contrary. Then $|x+y|=2$, and $\left|\frac{x+y}{2}\right|^p=1$. Since $p>1$, the function $g(t)=|t|^p$ is strictly convex, in other words $\left|\frac{x+y}{2}\right|^p<\frac{|x|^p+|y|^p}{2}$ whenever $x\neq y$. So for some $(x,y)\in D_\delta$ we have $\left|\frac{x+y}{2}\right|^p<\frac{|x|^p+|y|^p}{2}=1=\left|\frac{x+y}{2}\right|^p$. We get a contradiction.

If x and y have different signs then $(x,y)\in D_\delta$ for all $0<\delta<1$ because then $|x-y|\ge\max\{|x|,|y|\}\ge 1>\delta$. So we may further assume without loss of generality that $x>0$, $y>0$ and $x^p+y^p=2$. Set $x=1+t$. Then

$$\begin{aligned}
y &= (2-x^p)^{1/p}=(2-(1+t)^p)^{1/p}=\left(2-(1+pt+\frac{p(p-1)}{2}t^2+o(t^2))\right)^{1/p}\\
&= \left(1-pt-\frac{p(p-1)}{2}t^2+o(t^2)\right)^{1/p}\\
&= 1+\frac{1}{p}\left(-pt-\frac{p(p-1)}{2}t^2+o(t^2)\right)+\frac{1}{2p}\left(\frac{1}{p}-1\right)(-pt+o(t))^2+o(t^2)\\
&= 1-t-\frac{p-1}{2}t^2+o(t^2)-\frac{p-1}{2}t^2+o(t^2)\\
&= 1-t-(p-1)t^2+o(t^2).
\end{aligned}$$

We have

$$(x-y)^2=(2t+o(t))^2=4t^2+o(t^2)$$

and

$$4-(x+y)^2=4-(2-(p-1)t^2+o(t^2))^2=4-4+4(p-1)t^2+o(t^2)=4(p-1)t^2+o(t^2).$$

So there exists $\delta_p>0$ such that if $|t|<\delta_p$ we have $(x-y)^2<5t^2$, $4-(x+y)^2>3(p-1)t^2$. Then

$$(*)\qquad (x-y)^2<5t^2=\frac{5}{3(p-1)}\cdot 3(p-1)t^2<\frac{5}{3(p-1)}(4-(x+y)^2)$$

if $|x-1| < \delta_p$. From the symmetry we have that $(*)$ also holds when $|y-1| < \delta_p$.

To finish the proof it is enough to show that $|x-y| \geq 2\delta_p$ whenever $|x-1| \geq \delta_p$, $|y-1| \geq \delta_p$ and $x^p + y^p = 2$. Indeed, since $x^p + y^p = 2$ we have that $\max\{x, y\} \geq 1$. So let $x - 1 \geq \delta_p$. Since $\left(\frac{x+y}{2}\right)^p \leq \frac{x^p+y^p}{2} = 1$ we get $x + y \leq 2$. Then $x - y \geq 2(x-1) \geq 2\delta_p$.

Second day

Problem 1. (10 points)

Let A be 3×3 real matrix such that the vectors Au and u are orthogonal for each column vector $u \in \mathbb{R}^3$. Prove that:

a) $A^\top = -A$, where $A^\top$ denotes the transpose of the matrix A;

b) there exists a vector $v \in \mathbb{R}^3$ such that $Au = v \times u$ for every $u \in \mathbb{R}^3$, where $v \times u$ denotes the vector product in $\mathbb{R}^3$.

Solution. a) Set $A = (a_{ij})$, $u = (u_1, u_2, u_3)^\top$. If we use the orthogonality condition

$$(Au, u) = 0 \tag{1}$$

with $u_i = \delta_{ik}$ we get $a_{kk} = 0$. If we use (1) with $u_i = \delta_{ik} + \delta_{im}$ we get

$$a_{kk} + a_{km} + a_{mk} + a_{mm} = 0$$

and hence $a_{km} = -a_{mk}$.

b) Set $v_1 = -a_{23}$, $v_2 = a_{13}$, $v_3 = -a_{12}$. Then

$$Au = (v_2u_3 - v_3u_2, v_3u_1 - v_1u_3, v_1u_2 - v_2u_1)^\top = v \times u.$$

Problem 2. (15 points)

Let $\{b_n\}_{n=0}^\infty$ be a sequence of positive real numbers such that $b_0 = 1$, $b_n = 2 + \sqrt{b_{n-1}} - 2\sqrt{1 + \sqrt{b_{n-1}}}$. Calculate

$$\sum_{n=1}^\infty b_n 2^n.$$

Solution. Put $a_n = 1 + \sqrt{b_n}$ for $n \geq 0$. Then $a_n > 1$, $a_0 = 2$ and

$$a_n = 1 + \sqrt{1 + a_{n-1} - 2\sqrt{a_{n-1}}} = \sqrt{a_{n-1}},$$

so $a_n = 2^{2^{-n}}$. Then

$$\begin{aligned}
\sum_{n=1}^{N} b_n 2^n &= \sum_{n=1}^{N} (a_n - 1)^2 2^n = \sum_{n=1}^{N} [a_n^2 2^n - a_n 2^{n+1} + 2^n] \\
&= \sum_{n=1}^{N} [(a_{n-1} - 1)2^n - (a_n - 1)2^{n+1}] \\
&= (a_0 - 1)2^1 - (a_N - 1)2^{N+1} = 2 - 2\frac{2^{2^{-N}} - 1}{2^{-N}}.
\end{aligned}$$

Put $x = 2^{-N}$. Then $x \to 0$ as $N \to \infty$ and so

$$\sum_{n=1}^{\infty} b_n 2^N = \lim_{N\to\infty} \left(2 - 2\frac{2^{2^{-N}} - 1}{2^{-N}} \right) = \lim_{x\to 0} \left(2 - 2\frac{2^x - 1}{x} \right) = 2 - 2\ln 2.$$

Problem 3. (15 points)

Let all roots of an n-th degree polynomial $P(z)$ with complex coefficients lie on the unit circle in the complex plane. Prove that all roots of the polynomial

$$2zP'(z) - nP(z)$$

lie on the same circle.

Solution. It is enough to consider only polynomials with leading coefficient 1. Let $P(z) = (z - \alpha_1)(z - \alpha_2)\dots(z - \alpha_n)$ with $|\alpha_j| = 1$, where the complex numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ may coincide.

We have

$$\begin{aligned}
\widetilde{P}(z) \equiv\ & 2zP'(z) - nP(z) = (z + \alpha_1)(z - \alpha_2)\dots(z - \alpha_n) + \\
& +(z - \alpha_1)(z + \alpha_2)\dots(z - \alpha_n) + \dots + (z - \alpha_1)(z - \alpha_2)\dots(z + \alpha_n).
\end{aligned}$$

Hence, $\dfrac{\widetilde{P}(z)}{P(z)} = \displaystyle\sum_{k=1}^{n} \frac{z + \alpha_k}{z - \alpha_k}$. Since $\operatorname{Re}\dfrac{z + \alpha}{z - \alpha} = \dfrac{|z|^2 - |\alpha|^2}{|z - \alpha|^2}$ for all complex z, α, $z \neq \alpha$, we deduce that in our case $\operatorname{Re}\dfrac{\widetilde{P}(z)}{P(z)} = \displaystyle\sum_{k=1}^{n} \frac{|z|^2 - 1}{|z - \alpha_k|^2}$. From $|z| \neq 1$ it follows that $\operatorname{Re}\dfrac{\widetilde{P}(z)}{P(z)} \neq 0$. Hence $\widetilde{P}(z) = 0$ implies $|z| = 1$.

Problem 4. (15 points)

a) Prove that for every $\varepsilon > 0$ there is a positive integer n and real numbers $\lambda_1, \dots, \lambda_n$ such that

$$\max_{x\in[-1,1]} \left| x - \sum_{k=1}^{n} \lambda_k x^{2k+1} \right| < \varepsilon.$$

b) Prove that for every odd continuous function f on $[-1,1]$ and for every $\varepsilon > 0$ there is a positive integer n and real numbers $\mu_1, \dots, \mu_n$ such that

$$\max_{x\in[-1,1]} \left| f(x) - \sum_{k=1}^{n} \mu_k x^{2k+1} \right| < \varepsilon.$$

Recall that f is odd means that $f(x) = -f(-x)$ for all $x \in [-1,1]$.

Solution. a) Let n be such that $(1-\varepsilon^2)^n \leq \varepsilon$. Then $|x(1-x^2)^n| < \varepsilon$ for every $x \in [-1,1]$. Thus one can set $\lambda_k = (-1)^{k+1}\binom{n}{k}$ because then

$$x - \sum_{k=1}^{n} \lambda_k x^{2k+1} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{2k+1} = x(1-x^2)^n.$$

b) From the Weierstrass theorem there is a polynomial, say $p \in \Pi_m$, such that

$$\max_{x\in[-1,1]} |f(x) - p(x)| < \frac{\varepsilon}{2}.$$

Set $q(x) = \frac{1}{2}\{p(x) - p(-x)\}$. Then

$$f(x) - q(x) = \frac{1}{2}\{f(x) - p(x)\} - \frac{1}{2}\{f(-x) - p(-x)\}$$

and

$$(1)\quad \max_{|x|\leq 1} |f(x) - q(x)| \leq \frac{1}{2}\max_{|x|\leq 1} |f(x) - p(x)| + \frac{1}{2}\max_{|x|\leq 1} |f(-x) - p(-x)| < \frac{\varepsilon}{2}.$$

But q is an odd polynomial in Π_m and it can be written as

$$q(x) = \sum_{k=0}^{m} b_k x^{2k+1} = b_0 x + \sum_{k=1}^{m} b_k x^{2k+1}.$$

If $b_0 = 0$ then (1) proves b). If $b_0 \neq 0$ then one applies a) with $\dfrac{\varepsilon}{2|b_0|}$ instead of ε to get

$$\max_{|x|\leq 1}\left|b_0 x - \sum_{k=1}^{n} b_0 \lambda_k x^{2k+1}\right| < \frac{\varepsilon}{2} \tag{2}$$

for appropriate n and $\lambda_1, \lambda_2, \ldots, \lambda_n$. Now b) follows from (1) and (2) with $\max\{n, m\}$ instead of n.

Problem 5. (10+15 points)
a) Prove that every function of the form

$$f(x) = \frac{a_0}{2} + \cos x + \sum_{n=2}^{N} a_n \cos(nx)$$

with $|a_0| < 1$, has positive as well as negative values in the period $[0, 2\pi)$.
b) Prove that the function

$$F(x) = \sum_{n=1}^{100} \cos(n^{\frac{3}{2}} x)$$

has at least 40 zeros in the interval $(0, 1000)$.

Solution. a) Let us consider the integral

$$\int_0^{2\pi} f(x)(1 \pm \cos x)dx = \pi(a_0 \pm 1).$$

The assumption that $f(x) \geq 0$ implies $a_0 \geq 1$. Similarly, if $f(x) \leq 0$ then $a_0 \leq -1$. In both cases we have a contradiction with the hypothesis of the problem.

b) We shall prove that for each integer N and for each real number $h \geq 24$ and each real number y the function

$$F_N(x) = \sum_{n=1}^{N} \cos(x n^{\frac{3}{2}})$$

changes sign in the interval $(y, y+h)$. The assertion will follow immediately from here.

Consider the integrals

$$I_1 = \int_y^{y+h} F_N(x)dx, \qquad I_2 = \int_y^{y+h} F_N(x)\cos x\, dx.$$

If $F_N(x)$ does not change sign in $(y, y+h)$ then we have

$$|I_2| \leq \int_y^{y+h} |F_N(x)|dx = \left|\int_y^{y+h} F_N(x)dx\right| = |I_1|.$$

Hence, it is enough to prove that

$$|I_2| > |I_1|.$$

Obviously, for each $\alpha \neq 0$ we have

$$\left|\int_y^{y+h} \cos{(\alpha x)}dx\right| \leq \frac{2}{|\alpha|}.$$

Hence

$$|I_1| = \left|\sum_{n=1}^{N}\int_y^{y+h} \cos{(xn^{\frac{3}{2}})}dx\right| \leq 2\sum_{n=1}^{N}\frac{1}{n^{\frac{3}{2}}} < 2\left(1+\int_1^{\infty}\frac{dt}{t^{\frac{3}{2}}}\right) = 6. \tag{1}$$

On the other hand we have

$$\begin{aligned}
I_2 &= \sum_{n=1}^{N}\int_y^{y+h} \cos x \cos{(xn^{\frac{3}{2}})}dx \\
&= \frac{1}{2}\int_y^{y+h}(1+\cos{(2x)})dx + \\
&\quad +\frac{1}{2}\sum_{n=2}^{N}\int_y^{y+h}\left(\cos\left(x(n^{\frac{3}{2}}-1)\right)+\cos\left(x(n^{\frac{3}{2}}+1)\right)\right)dx \\
&= \frac{1}{2}h + \Delta,
\end{aligned}$$

where

$$|\Delta| \leq \frac{1}{2}\left(1+2\sum_{n=2}^{N}\left(\frac{1}{n^{\frac{3}{2}}-1}+\frac{1}{n^{\frac{3}{2}}+1}\right)\right) \leq \frac{1}{2}+2\sum_{n=2}^{N}\frac{1}{n^{\frac{3}{2}}-1}.$$

We use that $n^{\frac{3}{2}} - 1 \geq \frac{2}{3} n^{\frac{3}{2}}$ for $n \geq 3$ and we get

$$|\Delta| \leq \frac{1}{2} + \frac{2}{2^{\frac{3}{2}} - 1} + 3 \sum_{n=3}^{N} \frac{1}{n^{\frac{3}{2}}} < \frac{1}{2} + \frac{2}{2\sqrt{2} - 1} + 3 \int_2^\infty \frac{dt}{t^{\frac{3}{2}}} < 6.$$

Hence

$$|I_2| > \frac{1}{2} h - 6. \tag{2}$$

We use that $h \geq 24$ and inequalities (1), (2) and we obtain $|I_2| > |I_1|$. The proof is completed.

Problem 6. (20 points)

Suppose that $\{f_n\}_{n=1}^\infty$ is a sequence of continuous functions on the interval $[0, 1]$ such that

$$\int_0^1 f_m(x) f_n(x) dx = \begin{cases} 1 & \text{if} \quad n = m \\ 0 & \text{if} \quad n \neq m \end{cases}$$

and

$$\sup\{|f_n(x)| \ : \ x \in [0, 1] \text{ and } n = 1, 2, \ldots\} < +\infty.$$

Show that there exists no subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\lim\limits_{k \to \infty} f_{n_k}(x)$ exists for all $x \in [0, 1]$.

Solution. It is clear that one can add some functions, say $\{g_m\}$, which satisfy the hypothesis of the problem and the closure of the finite linear combinations of $\{f_n\} \cup \{g_m\}$ is $L_2[0, 1]$. Therefore without loss of generality we assume that $\{f_n\}$ generates $L_2[0, 1]$.

Let us suppose that there is a subsequence $\{n_k\}$ and a function f such that

$$f_{n_k}(x) \underset{k \to \infty}{\longrightarrow} f(x) \quad \text{for every} \quad x \in [0, 1].$$

Fix $m \in \mathbb{N}$. From Lebesgue's theorem we have

$$0 = \int_0^1 f_m(x) f_{n_k}(x) dx \underset{k \to \infty}{\longrightarrow} \int_0^1 f_m(x) f(x) dx.$$

Hence $\int_0^1 f_m(x) f(x) dx = 0$ for every $m \in \mathbb{N}$, which implies $f(x) = 0$ almost everywhere. Using once more Lebesgue's theorem we get

$$1 = \int_0^1 f_{n_k}^2(x) dx \underset{k \to \infty}{\longrightarrow} \int_0^1 f^2(x) dx = 0.$$

The contradiction proves the statement.