FOURTH INTERNATIONAL COMPETITION FOR UNIVERSITY STUDENTS IN MATHEMATICS
July 30 – August 4, 1997, Plovdiv, BULGARIA

First day — August 1, 1997

Problems and Solutions

Problem 1.

Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers, such that $\lim_{n\to\infty} \varepsilon_n = 0$. Find

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left(\frac{k}{n} + \varepsilon_n\right),$$

where ln denotes the natural logarithm.

Solution.

It is well known that

$$-1 = \int_0^1 \ln x dx = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left(\frac{k}{n}\right)$$

(Riemman's sums). Then

$$\frac{1}{n} \sum_{k=1}^{n} \ln\left(\frac{k}{n} + \varepsilon_n\right) \geq \frac{1}{n} \sum_{k=1}^{n} \ln\left(\frac{k}{n}\right) \underset{n\to\infty}{\longrightarrow} -1.$$

Given $\varepsilon > 0$ there exist n_0 such that $0 < \varepsilon_n \leq \varepsilon$ for all $n \geq n_0$. Then

$$\frac{1}{n} \sum_{k=1}^{n} \ln\left(\frac{k}{n} + \varepsilon_n\right) \leq \frac{1}{n} \sum_{k=1}^{n} \ln\left(\frac{k}{n} + \varepsilon\right).$$

Since

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left(\frac{k}{n} + \varepsilon\right) &= \int_0^1 \ln(x+\varepsilon) dx \\ &= \int_{\varepsilon}^{1+\varepsilon} \ln x dx \end{aligned}$$

we obtain the result when ε goes to 0 and so

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left(\frac{k}{n} + \varepsilon_n\right) = -1.$$

Problem 2.

Suppose $\sum\limits_{n=1}^{\infty} a_n$ converges. Do the following sums have to converge as well?

a) $a_1 + a_2 + a_4 + a_3 + a_8 + a_7 + a_6 + a_5 + a_{16} + a_{15} + \cdots + a_9 + a_{32} + \cdots$

b) $a_1 + a_2 + a_3 + a_4 + a_5 + a_7 + a_6 + a_8 + a_9 + a_{11} + a_{13} + a_{15} + a_{10} + a_{12} + a_{14} + a_{16} + a_{17} + a_{19} + \cdots$

Justify your answers.

Solution.

a) Yes. Let $S = \sum\limits_{n=1}^{\infty} a_n$, $S_n = \sum\limits_{k=1}^{n} a_k$. Fix $\varepsilon > 0$ and a number n_0 such that $|S_n - S| < \varepsilon$ for $n > n_0$. The partial sums of the permuted series have the form $L_{2^{n-1}+k} = S_{2^{n-1}} + S_{2^n} - S_{2^n-k}$, $0 \le k < 2^{n-1}$ and for $2^{n-1} > n_0$ we have $|L_{2^{n-1}+k} - S| < 3\varepsilon$, i.e. the permuted series converges.

b) No. Take $a_n = \dfrac{(-1)^{n+1}}{\sqrt{n}}$.Then $L_{3.2^{n-2}} = S_{2^{n-1}} + \sum\limits_{k=2^{n-2}}^{2^{n-1}-1} \dfrac{1}{\sqrt{2k+1}}$ and $L_{3.2^{n-2}} - S_{2^{n-1}} \ge 2^{n-2} \dfrac{1}{\sqrt{2^n}} \underset{n\to\infty}{\longrightarrow} \infty$, so $L_{3.2^{n-2}} \underset{n\to\infty}{\longrightarrow} \infty$.

Problem 3.

Let A and B be real $n\times n$ matrices such that $A^2+B^2=AB$. Prove that if $BA - AB$ is an invertible matrix then n is divisible by 3.

Solution.

Set $S = A + \omega B$, where $\omega = -\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$. We have

$$\begin{aligned} S\overline{S} &= (A + \omega B)(A + \overline{\omega} B) = A^2 + \omega BA + \overline{\omega} AB + B^2 \\ &= AB + \omega BA + \overline{\omega} AB = \omega(BA - AB), \end{aligned}$$

because $\overline{\omega} + 1 = -\omega$. Since $\det(S\overline{S}) = \det S. \det \overline{S}$ is a real number and $\det \omega(BA - AB) = \omega^n \det(BA - AB)$ and $\det(BA - AB) \neq 0$, then ω^n is a real number. This is possible only when n is divisible by 3.

Problem 4.

Let α be a real number, $1 < \alpha < 2$.

a) Show that α has a unique representation as an infinite product

$$\alpha = \left(1 + \frac{1}{n_1}\right)\left(1 + \frac{1}{n_2}\right)\ldots$$

where each n_i is a positive integer satisfying

$$n_i^2 \leq n_{i+1}.$$

b) Show that α is rational if and only if its infinite product has the following property:

For some m and all $k \geq m$,

$$n_{k+1} = n_k^2.$$

Solution.

a) We construct inductively the sequence $\{n_i\}$ and the ratios

$$\theta_k = \frac{\alpha}{\prod_1^k (1 + \frac{1}{n_i})}$$

so that

$$\theta_k > 1 \text{ for all } k.$$

Choose n_k to be the least n for which

$$1 + \frac{1}{n} < \theta_{k-1}$$

($\theta_0 = \alpha$) so that for each k,

(1) $$1 + \frac{1}{n_k} < \theta_{k-1} \leq 1 + \frac{1}{n_k - 1}.$$

Since

$$\theta_{k-1} \leq 1 + \frac{1}{n_k - 1}$$

we have

$$1 + \frac{1}{n_{k+1}} < \theta_k = \frac{\theta_{k-1}}{1 + \frac{1}{n_k}} \leq \frac{1 + \frac{1}{n_k - 1}}{1 + \frac{1}{n_k}} = 1 + \frac{1}{n_k^2 - 1}.$$

Hence, for each k, $n_{k+1} \geq n_k^2$.

Since $n_1 \geq 2$, $n_k \to \infty$ so that $\theta_k \to 1$. Hence

$$\alpha = \prod_1^\infty \left(1 + \frac{1}{n_k}\right).$$

The uniquness of the infinite product will follow from the fact that on every step n_k has to be determine by (1).

Indeed, if for some k we have

$$1 + \frac{1}{n_k} \geq \theta_{k-1}$$

then $\theta_k \leq 1$, $\theta_{k+1} < 1$ and hence $\{\theta_k\}$ does not converge to 1.

Now observe that for $M > 1$,

$$(2) \quad \left(1 + \frac{1}{M}\right)\left(1 + \frac{1}{M^2}\right)\left(1 + \frac{1}{M^4}\right) \cdots = 1 + \frac{1}{M} + \frac{1}{M^2} + \frac{1}{M^3} + \cdots = 1 + \frac{1}{M-1}.$$

Assume that for some k we have

$$1 + \frac{1}{n_k - 1} < \theta_{k-1}.$$

Then we get

$$\begin{aligned} \frac{\alpha}{(1+\frac{1}{n_1})(1+\frac{1}{n_2})\ldots} &= \frac{\theta_{k-1}}{(1+\frac{1}{n_k})(1+\frac{1}{n_{k+1}})\ldots} \\ &\geq \frac{\theta_{k-1}}{(1+\frac{1}{n_k})(1+\frac{1}{n_k^2})\ldots} = \frac{\theta_{k-1}}{1+\frac{1}{n_k-1}} > 1 \end{aligned}$$

– a contradiction.

b) From (2) α is rational if its product ends in the stated way.

Conversely, suppose α is the rational number $\frac{p}{q}$. Our aim is to show that for some m,

$$\theta_{m-1} = \frac{n_m}{n_m - 1}.$$

Suppose this is not the case, so that for every m,

$$(3) \qquad \theta_{m-1} < \frac{n_m}{n_m - 1}.$$

For each k we write

$$\theta_k = \frac{p_k}{q_k}$$

as a fraction (not necessarily in lowest terms) where

$$p_0 = p, \quad q_0 = q$$

and in general

$$p_k = p_{k-1} n_k, \quad q_k = q_{k-1}(n_k + 1).$$

The numbers $p_k - q_k$ are positive integers: to obtain a contradiction it suffices to show that this sequence is strictly decreasing. Now,

$$\begin{aligned} p_k - q_k - (p_{k-1} - q_{k-1}) &= n_k p_{k-1} - (n_k + 1) q_{k-1} - p_{k-1} + q_{k-1} \\ &= (n_k - 1) p_{k-1} - n_k q_{k-1} \end{aligned}$$

and this is negative because

$$\frac{p_{k-1}}{q_{k-1}} = \theta_{k-1} < \frac{n_k}{n_k - 1}$$

by inequality (3).

Problem 5. For a natural n consider the hyperplane

$$R_0^n = \left\{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 0 \right\}$$

and the lattice $Z_0^n = \{y \in R_0^n : \text{ all } y_i \text{ are integers}\}$. Define the (quasi–)norm in $\mathbb{R}^n$ by $\|x\|_p = \left(\sum\limits_{i=1}^{n} |x_i|^p \right)^{1/p}$ if $0 < p < \infty$, and $\|x\|_\infty = \max\limits_i |x_i|$.

a) Let $x \in R_0^n$ be such that

$$\max_i x_i - \min_i x_i \le 1.$$

For every $p \in [1, \infty]$ and for every $y \in Z_0^n$ prove that

$$\|x\|_p \le \|x + y\|_p.$$

b) For every $p \in (0, 1)$, show that there is an n and an $x \in R_0^n$ with $\max\limits_i x_i - \min\limits_i x_i \le 1$ and an $y \in Z_0^n$ such that

$$\|x\|_p > \|x + y\|_p.$$

Solution.

a) For $x = 0$ the statement is trivial. Let $x \neq 0$. Then $\max_i x_i > 0$ and $\min_i x_i < 0$. Hence $\|x\|_\infty < 1$. From the hypothesis on x it follows that:

i) If $x_j \leq 0$ then $\max_i x_i \leq x_j + 1$.

ii) If $x_j \geq 0$ then $\min_i x_i \geq x_j - 1$.

Consider $y \in Z_0^n$, $y \neq 0$. We split the indices $\{1, 2, \ldots, n\}$ into five sets:

$$I(0) = \{i : y_i = 0\},$$

$$I(+,+) = \{i : y_i > 0, x_i \geq 0\}, \quad I(+,-) = \{i : y_i > 0, x_i < 0\},$$

$$I(-,+) = \{i : y_i < 0, x_i > 0\}, \quad I(-,-) = \{i : y_i < 0, x_i \leq 0\}.$$

As least one of the last four index sets is not empty. If $I(+,+) \neq \emptyset$ or $I(-,-) \neq \emptyset$ then $\|x+y\|_\infty \geq 1 > \|x\|_\infty$. If $I(+,+) = I(-,-) = \emptyset$ then $\sum y_i = 0$ implies $I(+,-) \neq \emptyset$ and $I(-,+) \neq \emptyset$. Therefore i) and ii) give $\|x+y\|_\infty \geq \|x\|_\infty$ which completes the case $p = \infty$.

Now let $1 \leq p < \infty$. Then using i) for every $j \in I(+,-)$ we get $|x_j + y_j| = y_j - 1 + x_j + 1 \geq |y_j| - 1 + \max_i x_i$. Hence

$$|x_j + y_j|^p \geq |y_j| - 1 + |x_k|^p \quad \text{for every } k \in I(-,+) \quad \text{and} \quad j \in I(+,-).$$

Similarly

$$|x_j + y_j|^p \geq |y_j| - 1 + |x_k|^p \quad \text{for every } k \in I(+,-) \quad \text{and} \quad j \in I(-,+);$$

$$|x_j + y_j|^p \geq |y_j| + |x_j|^p \quad \text{for every } j \in I(+,+) \cup I(-,-).$$

Assume that $\sum_{j \in I(+,-)} 1 \geq \sum_{j \in I(-,+)} 1$. Then

$$\begin{aligned}
&\|x+y\|_p^p - \|x\|_p^p \\
&= \sum_{j \in I(+,+) \cup I(-,-)} \left(|x_j + y_j|^p - |x_j|^p\right) + \left(\sum_{j \in I(+,-)} |x_j + y_j|^p - \sum_{k \in I(-,+)} |x_k|^p \right) \\
&\quad + \left(\sum_{j \in I(-,+)} |x_j + y_j|^p - \sum_{k \in I(+,-)} |x_k|^p \right) \\
&\geq \sum_{j \in I(+,+) \cup I(-,-)} |y_j| + \sum_{j \in I(+,-)} (|y_j| - 1)
\end{aligned}$$

$$+\left(\sum_{j\in I(-,+)}(|y_j|-1)-\sum_{j\in I(+,-)}1+\sum_{j\in I(-,+)}1\right)$$

$$=\sum_{i=1}^{n}|y_i|-2\sum_{j\in I(+,-)}1=2\sum_{j\in I(+,-)}(y_j-1)+2\sum_{j\in I(+,+)}y_j\geq 0.$$

The case $\sum_{j\in I(+,-)} 1 \leq \sum_{j\in I(-,+)} 1$ is similar. This proves the statement.

b) Fix $p \in (0,1)$ and a rational $t \in (\frac{1}{2}, 1)$. Choose a pair of positive integers m and l such that $mt = l(1-t)$ and set $n = m + l$. Let

$$x_i = t, \quad i = 1,2,\ldots,m; \quad x_i = t-1, \quad i = m+1, m+2, \ldots, n;$$

$$y_i = -1, \quad i = 1,2,\ldots,m; \quad y_{m+1} = m; \quad y_i = 0, \ i = m+2,\ldots,n.$$

Then $x \in R_0^n$, $\max_i x_i - \min_i x_i = 1$, $y \in Z_0^n$ and

$$\|x\|_p^p - \|x+y\|_p^p = m(t^p - (1-t)^p) + (1-t)^p - (m-1+t)^p,$$

which is possitive for m big enough.

Problem 6. Suppose that F is a family of finite subsets of $\mathbb{N}$ and for any two sets $A, B \in F$ we have $A \cap B \neq \emptyset$.

a) Is it true that there is a finite subset Y of $\mathbb{N}$ such that for any $A, B \in F$ we have $A \cap B \cap Y \neq \emptyset$?

b) Is the statement a) true if we suppose in addition that all of the members of F have the same size?

Justify your answers.

Solution.

a) No. Consider $F = \{A_1, B_1, \ldots, A_n, B_n, \ldots\}$, where $A_n = \{1,3,5,\ldots,2n-1, 2n\}$, $B_n = \{2,4,6,\ldots,2n,2n+1\}$.

b) Yes. We will prove inductively a stronger statement:

Suppose F, G are two families of finite subsets of $\mathbb{N}$ such that:

1) For every $A \in F$ and $B \in G$ we have $A \cap B \neq \emptyset$;

2) All the elements of F have the same size r, and elements of G – size s. (we shall write $\#(F) = r$, $\#(G) = s$).

Then there is a finite set Y such that $A \cup B \cup Y \neq \emptyset$ for every $A \in F$ and $B \in G$.

The problem b) follows if we take $F = G$.

Proof of the statement: The statement is obvious for $r = s = 1$. Fix the numbers r, s and suppose the statement is proved for all pairs F', G' with $\#(F') < r$, $\#(G') < s$. Fix $A_0 \in F$, $B_0 \in G$. For any subset $C \subset A_0 \cup B_0$, denote

$$F(C) = \{A \in F : A \cap (A_0 \cup B_0) = C\}.$$

Then $F = \bigcup_{\emptyset \neq C \subset A_0 \cup B_0} F(C)$. It is enough to prove that for any pair of non-empty sets $C, D \subset A_0 \cup B_0$ the families $F(C)$ and $G(D)$ satisfy the statement.

Indeed, if we denote by $Y_{C,D}$ the corresponding finite set, then the finite set $\bigcup_{C,D \subset A_0 \cup B_0} Y_{C,D}$ will satisfy the statement for F and G. The proof for $F(C)$ and $G(D)$.

If $C \cap D \neq \emptyset$, it is trivial.

If $C \cap D = \emptyset$, then any two sets $A \in F(C)$, $B \in G(D)$ must meet outside $A_0 \cup B_0$. Then if we denote $\tilde{F}(C) = \{A \setminus C : A \in F(C)\}$, $\tilde{G}(D) = \{B \setminus D : B \in G(D)\}$, then $\tilde{F}(C)$ and $\tilde{G}(D)$ satisfy the conditions 1) and 2) above, with $\#(\tilde{F}(C)) = \#(F) - \#C < r$, $\#(\tilde{G}(D)) = \#(G) - \#D < s$, and the inductive assumption works.