FOURTH INTERNATIONAL COMPETITION FOR UNIVERSITY STUDENTS IN MATHEMATICS July 30 – August 4, 1997, Plovdiv, BULGARIA

Second day — August 2, 1997

Problems and Solutions

Problem 1.

Let f be a $C^3(\mathbb{R})$ non-negative function, $f(0)=f'(0)=0$, $0 < f''(0)$. Let

$$g(x) = \left(\frac{\sqrt{f(x)}}{f'(x)}\right)'$$

for $x \neq 0$ and $g(0) = 0$. Show that g is bounded in some neighbourhood of 0. Does the theorem hold for $f \in C^2(\mathbb{R})$?

Solution.

Let $c = \dfrac{1}{2}f''(0)$. We have

$$g = \frac{(f')^2 - 2ff''}{2(f')^2\sqrt{f}},$$

where

$$f(x) = cx^2 + O(x^3), \quad f'(x) = 2cx + O(x^2), \quad f''(x) = 2c + O(x).$$

Therefore $(f'(x))^2 = 4c^2x^2 + O(x^3)$,

$$2f(x)f''(x) = 4c^2x^2 + O(x^3)$$

and

$$2(f'(x))^2\sqrt{f(x)} = 2(4c^2x^2 + O(x^3))|x|\sqrt{c + O(x)}.$$

g is bounded because

$$\frac{2(f'(x))^2\sqrt{f(x)}}{|x|^3} \underset{x\to 0}{\longrightarrow} 8c^{5/2} \neq 0$$

and $f'(x)^2 - 2f(x)f''(x) = O(x^3)$.

The theorem does not hold for some C^2-functions.

Let $f(x) = (x + |x|^{3/2})^2 = x^2 + 2x^2\sqrt{|x|} + |x|^3$, so f is C^2. For $x > 0$,

$$g(x) = \frac{1}{2}\left(\frac{1}{1+\frac{3}{2}\sqrt{x}}\right)' = -\frac{1}{2} \cdot \frac{1}{(1+\frac{3}{2}\sqrt{x})^2} \cdot \frac{3}{4} \cdot \frac{1}{\sqrt{x}} \underset{x\to 0}{\longrightarrow} -\infty.$$

Problem 2.

Let M be an invertible matrix of dimension $2n \times 2n$, represented in block form as

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{and} \quad M^{-1} = \begin{bmatrix} E & F \\ G & H \end{bmatrix}.$$

Show that $\det M . \det H = \det A$.

Solution.

Let I denote the identity $n \times n$ matrix. Then

$$\det M . \det H = \det \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \det \begin{bmatrix} I & F \\ 0 & H \end{bmatrix} = \det \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} = \det A.$$

Problem 3.

Show that $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}\sin(\log n)}{n^\alpha}$ converges if and only if $\alpha > 0$.

Solution.

Set $f(t) = \dfrac{\sin(\log t)}{t^\alpha}$. We have

$$f'(t) = \frac{-\alpha}{t^{\alpha+1}}\sin(\log t) + \frac{\cos(\log t)}{t^{\alpha+1}}.$$

So $|f'(t)| \leq \dfrac{1+\alpha}{t^{\alpha+1}}$ for $\alpha > 0$. Then from Mean value theorem for some $\theta \in (0,1)$ we get $|f(n+1) - f(n)| = |f'(n+\theta)| \leq \dfrac{1+\alpha}{n^{\alpha+1}}$. Since $\sum \dfrac{1+\alpha}{n^{\alpha+1}} < +\infty$ for $\alpha > 0$ and $f(n) \underset{n\to\infty}{\longrightarrow} 0$ we get that $\sum\limits_{n=1}^{\infty} (-1)^{n-1} f(n) = \sum\limits_{n=1}^{\infty} (f(2n-1) - f(2n))$ converges.

Now we have to prove that $\dfrac{\sin(\log n)}{n^\alpha}$ does not converge to 0 for $\alpha \leq 0$. It suffices to consider $\alpha = 0$.

We show that $a_n = \sin(\log n)$ does not tend to zero. Assume the contrary. There exist $k_n \in \mathbb{N}$ and $\lambda_n \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$ for $n > e^2$ such that $\dfrac{\log n}{\pi} = k_n + \lambda_n$. Then $|a_n| = \sin \pi |\lambda_n|$. Since $a_n \to 0$ we get $\lambda_n \to 0$.

We have $k_{n+1} - k_n =$

$$= \frac{\log(n+1) - \log n}{\pi} - (\lambda_{n+1} - \lambda_n) = \frac{1}{\pi}\log\left(1 + \frac{1}{n}\right) - (\lambda_{n+1} - \lambda_n).$$

Then $|k_{n+1} - k_n| < 1$ for all n big enough. Hence there exists n_0 so that $k_n = k_{n_0}$ for $n > n_0$. So $\frac{\log n}{\pi} = k_{n_0} + \lambda_n$ for $n > n_0$. Since $\lambda_n \to 0$ we get contradiction with $\log n \to \infty$.

Problem 4.

a) Let the mapping $f : M_n \to \mathbb{R}$ from the space $M_n = \mathbb{R}^{n^2}$ of $n \times n$ matrices with real entries to reals be linear, i.e.:

$$f(A+B) = f(A) + f(B), \quad f(cA) = cf(A) \tag{1}$$

for any $A, B \in M_n$, $c \in \mathbb{R}$. Prove that there exists a unique matrix $C \in M_n$ such that $f(A) = \operatorname{tr}(AC)$ for any $A \in M_n$. (If $A = \{a_{ij}\}_{i,j=1}^n$ then $\operatorname{tr}(A) = \sum_{i=1}^n a_{ii}$).

b) Suppose in addition to (1) that

$$f(A.B) = f(B.A) \tag{2}$$

for any $A, B \in M_n$. Prove that there exists $\lambda \in \mathbb{R}$ such that $f(A) = \lambda.\operatorname{tr}(A)$.

Solution.

a) If we denote by E_{ij} the standard basis of M_n consisting of elementary matrix (with entry 1 at the place (i,j) and zero elsewhere), then the entries c_{ij} of C can be defined by $c_{ij} = f(E_{ji})$. b) Denote by L the n^2-1-dimensional linear subspace of M_n consisting of all matrices with zero trace. The elements E_{ij} with $i \neq j$ and the elements $E_{ii} - E_{nn}$, $i = 1, \ldots, n-1$ form a linear basis for L. Since

$$\begin{aligned} E_{ij} &= E_{ij}.E_{jj} - E_{jj}.E_{ij}, \ i \neq j \\ E_{ii} - E_{nn} &= E_{in}.E_{ni} - E_{ni}.E_{in}, i = 1, \ldots, n-1, \end{aligned}$$

then the property (2) shows that f is vanishing identically on L. Now, for any $A \in M_n$ we have $A - \frac{1}{n}\operatorname{tr}(A).E \in L$, where E is the identity matrix, and therefore $f(A) = \frac{1}{n}f(E).\operatorname{tr}(A)$.

Problem 5.

Let X be an arbitrary set, let f be an one-to-one function mapping X onto itself. Prove that there exist mappings $g_1, g_2 : X \to X$ such that $f = g_1 \circ g_2$ and $g_1 \circ g_1 = id = g_2 \circ g_2$, where id denotes the identity mapping on X.

Solution.

Let $f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$, $f^0 = id$, $f^{-n} = (f^{-1})^n$ for every natural number n. Let $T(x) = \{f^n(x) : n \in \mathbb{Z}\}$ for every $x \in X$. The sets $T(x)$ for different x's either coinside or do not intersect. Each of them is mapped by f onto itself. It is enough to prove the theorem for every such set. Let $A = T(x)$. If A is finite, then we can think that A is the set of all vertices of a regular n polygon and that f is rotation by $\dfrac{2\pi}{n}$. Such rotation can be obtained as a composition of 2 symmetries mapping the n polygon onto itself (if n is even then there are axes of symmetry making $\dfrac{\pi}{n}$ angle; if $n = 2k + 1$ then there are axes making $k\dfrac{2\pi}{n}$ angle). If A is infinite then we can think that $A = \mathbb{Z}$ and $f(m) = m + 1$ for every $m \in \mathbb{Z}$. In this case we define g_1 as a symmetry relative to $\dfrac{1}{2}$, g_2 as a symmetry relative to 0.

Problem 6.

Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Say that f "crosses the axis" at x if $f(x) = 0$ but in any neighbourhood of x there are y, z with $f(y) < 0$ and $f(z) > 0$.

a) Give an example of a continuous function that "crosses the axis" infiniteley often.

b) Can a continuous function "cross the axis" uncountably often?

Justify your answer.

Solution.

a) $f(x) = x \sin \dfrac{1}{x}$.

b) Yes. The Cantor set is given by

$$C = \{x \in [0, 1) : x = \sum_{j=1}^{\infty} b_j 3^{-j},\ b_j \in \{0, 2\}\}.$$

There is an one-to-one mapping $f : [0, 1) \to C$. Indeed, for $x = \sum\limits_{j=1}^{\infty} a_j 2^{-j}$, $a_j \in \{0, 1\}$ we set $f(x) = \sum\limits_{j=1}^{\infty} (2a_j) 3^{-j}$. Hence C is uncountable.

For $k = 1, 2, \dots$ and $i = 0, 1, 2, \dots, 2^{k-1} - 1$ we set

$$a_{k,i} = 3^{-k} \left(6 \sum_{j=0}^{k-2} a_j 3^j + 1 \right), \quad b_{k,i} = 3^{-k} \left(6 \sum_{j=0}^{k-2} a_j 3^j + 2 \right),$$

where $i = \sum\limits_{j=0}^{k-2} a_j 2^j$, $a_j \in \{0, 1\}$. Then

$$[0,1) \setminus C = \bigcup_{k=1}^{\infty} \bigcup_{i=0}^{2^{k-1}-1} (a_{k,i}, b_{k,i}),$$

i.e. the Cantor set consists of all points which have a trinary representation with 0 and 2 as digits and the points of its compliment have some 1's in their trinary representation. Thus, $\bigcup\limits_{i=0}^{2^{k-1}-1} (a_{k,i}, b_{k,i})$ are all points (exept $a_{k,i}$) which have 1 on k-th place and 0 or 2 on the j-th ($j < k$) places.

Noticing that the points with at least one digit equals to 1 are everywhere dence in [0,1] we set

$$f(x) = \sum_{k=1}^{\infty} (-1)^k g_k(x).$$

where g_k is a piece-wise linear continuous functions with values at the knots $g_k\left(\dfrac{a_{k,i} + b_{k,i}}{2}\right) = 2^{-k}$, $g_k(0) = g_k(1) = g_k(a_{k,i}) = g_k(b_{k,i}) = 0$, $i = 0, 1, \dots, 2^{k-1} - 1$.

Then f is continuous and f "crosses the axis" at every point of the Cantor set.