8^{th} **IMC 2001**
July 19 - July 25
Prague, Czech Republic

First day

Problem 1.
Let n be a positive integer. Consider an $n \times n$ matrix with entries $1, 2, \ldots, n^2$ written in order starting top left and moving along each row in turn left–to–right. We choose n entries of the matrix such that exactly one entry is chosen in each row and each column. What are the possible values of the sum of the selected entries?

Solution. Since there are exactly n rows and n columns, the choice is of the form

$$\{(j, \sigma(j)) : j = 1, \ldots, n\}$$

where $\sigma \in S_n$ is a permutation. Thus the corresponding sum is equal to

$$\sum_{j=1}^{n} n(j-1) + \sigma(j) = \sum_{j=1}^{n} nj - \sum_{j=1}^{n} n + \sum_{j=1}^{n} \sigma(j)$$

$$= n \sum_{j=1}^{n} j - \sum_{j=1}^{n} n + \sum_{j=1}^{n} j = (n+1)\frac{n(n+1)}{2} - n^2 = \frac{n(n^2+1)}{2},$$

which shows that the sum is independent of σ.

Problem 2.
Let r, s, t be positive integers which are pairwise relatively prime. If a and b are elements of a commutative multiplicative group with unity element e, and $a^r = b^s = (ab)^t = e$, prove that $a = b = e$.
Does the same conclusion hold if a and b are elements of an arbitrary non-commutative group?

Solution. 1. There exist integers u and v such that $us + vt = 1$. Since $ab = ba$, we obtain

$$ab = (ab)^{us+vt} = (ab)^{us} \left((ab)^t\right)^v = (ab)^{us} e = (ab)^{us} = a^{us} (b^s)^u = a^{us} e = a^{us}.$$

Therefore, $b^r = eb^r = a^r b^r = (ab)^r = a^{usr} = (a^r)^{us} = e$. Since $xr + ys = 1$ for suitable integers x and y,

$$b = b^{xr+ys} = (b^r)^x (b^s)^y = e.$$

It follows similarly that $a = e$ as well.
2. This is not true. Let $a = (123)$ and $b = (34567)$ be cycles of the permutation group S_7 of order 7. Then $ab = (1234567)$ and $a^3 = b^5 = (ab)^7 = e$.

Problem 3. Find $\lim\limits_{t \nearrow 1} (1-t) \sum\limits_{n=1}^{\infty} \frac{t^n}{1+t^n}$, where $t \nearrow 1$ means that t approaches 1 from below.

Solution.

$$\lim_{t\to 1-0}(1-t)\sum_{n=1}^{\infty}\frac{t^n}{1+t^n}=\lim_{t\to 1-0}\frac{1-t}{-\ln t}\cdot(-\ln t)\sum_{n=1}^{\infty}\frac{t^n}{1+t^n}=$$

$$=\lim_{t\to 1-0}(-\ln t)\sum_{n=1}^{\infty}\frac{1}{1+e^{-n\ln t}}=\lim_{h\to+0}h\sum_{n=1}^{\infty}\frac{1}{1+e^{nh}}=\int_0^{\infty}\frac{dx}{1+e^x}=\ln 2.$$

Problem 4.

Let k be a positive integer. Let $p(x)$ be a polynomial of degree n each of whose coefficients is -1, 1 or 0, and which is divisible by $(x-1)^k$. Let q be a prime such that $\frac{q}{\ln q}<\frac{k}{\ln(n+1)}$. Prove that the complex qth roots of unity are roots of the polynomial $p(x)$.

Solution. Let $p(x)=(x-1)^k\cdot r(x)$ and $\varepsilon_j=e^{2\pi i\cdot j/q}$ $(j=1,2,\dots,q-1)$. As is well-known, the polynomial $x^{q-1}+x^{q-2}+\ldots+x+1=(x-\varepsilon_1)\ldots(x-\varepsilon_{q-1})$ is irreducible, thus all $\varepsilon_1,\dots,\varepsilon_{q-1}$ are roots of $r(x)$, or none of them.

Suppose that none of $\varepsilon_1,\dots,\varepsilon_{q-1}$ is a root of $r(x)$. Then $\prod_{j=1}^{q-1}r(\varepsilon_j)$ is a rational integer, which is not 0 and

$$(n+1)^{q-1}\ge\prod_{j=1}^{q-1}\left|p(\varepsilon_j)\right|=\left|\prod_{j=1}^{q-1}(1-\varepsilon_j)^k\right|\cdot\left|\prod_{j=1}^{q-1}r(\varepsilon_j)\right|\ge$$

$$\ge\left|\prod_{j=1}^{q-1}(1-\varepsilon_j)\right|^k=(1^{q-1}+1^{q-2}+\ldots+1^1+1)^k=q^k.$$

This contradicts the condition $\frac{q}{\ln q}<\frac{k}{\ln(n+1)}$.

Problem 5.

Let A be an $n\times n$ complex matrix such that $A\ne\lambda I$ for all $\lambda\in\mathbf{C}$. Prove that A is similar to a matrix having at most one non-zero entry on the main diagonal.

Solution. The statement will be proved by induction on n. For $n=1$, there is nothing to do. In the case $n=2$, write $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$. If $b\ne 0$, and $c\ne 0$ or $b=c=0$ then A is similar to

$$\begin{bmatrix}1&0\\a/b&1\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1&0\\-a/b&1\end{bmatrix}=\begin{bmatrix}0&b\\c-ad/b&a+d\end{bmatrix}$$

or

$$\begin{bmatrix}1&-a/c\\0&1\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1&a/c\\0&1\end{bmatrix}=\begin{bmatrix}0&b-ad/c\\c&a+d\end{bmatrix},$$

respectively. If $b=c=0$ and $a\ne d$, then A is similar to

$$\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}a&0\\0&d\end{bmatrix}\begin{bmatrix}1&-1\\0&1\end{bmatrix}=\begin{bmatrix}a&d-a\\0&d\end{bmatrix},$$

and we can perform the step seen in the case $b \neq 0$ again.

Assume now that $n > 3$ and the problem has been solved for all $n' < n$. Let $A = \begin{bmatrix} A' & * \\ * & \beta \end{bmatrix}_n$, where A' is $(n-1) \times (n-1)$ matrix. Clearly we may assume that $A' \neq \lambda' I$, so the induction provides a P with, say, $P^{-1}A'P = \begin{bmatrix} 0 & * \\ * & \alpha \end{bmatrix}_{n-1}$. But then the matrix

$$B = \begin{bmatrix} P^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A' & * \\ * & \beta \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} P^{-1}A'P & * \\ * & \beta \end{bmatrix}$$

is similar to A and its diagonal is $(0, 0, \dots, 0, \alpha, \beta)$. On the other hand, we may also view B as $\begin{bmatrix} 0 & * \\ * & C \end{bmatrix}_n$, where C is an $(n-1) \times (n-1)$ matrix with diagonal $(0, \dots, 0, \alpha, \beta)$. If the inductive hypothesis is applicable to C, we would have $Q^{-1}CQ = D$, with $D = \begin{bmatrix} 0 & * \\ * & \gamma \end{bmatrix}_{n-1}$ so that finally the matrix

$$E = \begin{bmatrix} 1 & 0 \\ 0 & Q^{-1} \end{bmatrix} \cdot B \cdot \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & Q^{-1} \end{bmatrix} \begin{bmatrix} 0 & * \\ * & C \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} 0 & * \\ * & D \end{bmatrix}$$

is similar to A and its diagonal is $(0, 0, \dots, 0, \gamma)$, as required.

The inductive argument can fail only when $n - 1 = 2$ and the resulting matrix applying P has the form

$$P^{-1}AP = \begin{bmatrix} 0 & a & b \\ c & d & 0 \\ e & 0 & d \end{bmatrix}$$

where $d \neq 0$. The numbers a, b, c, e cannot be 0 at the same time. If, say, $b \neq 0$, A is similar to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & a & b \\ c & d & 0 \\ e & 0 & d \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -b & a & b \\ c & d & 0 \\ e-b-d & a & b+d \end{bmatrix}.$$

Performing half of the induction step again, the diagonal of the resulting matrix will be $(0, d-b, d+b)$ (the trace is the same) and the induction step can be finished. The cases $a \neq 0$, $c \neq 0$ and $e \neq 0$ are similar.

Problem 6.

Suppose that the differentiable functions $a, b, f, g : \mathbb{R} \to \mathbb{R}$ satisfy

$$f(x) \geq 0, f'(x) \geq 0, g(x) > 0, g'(x) > 0 \text{ for all } x \in \mathbb{R},$$

$$\lim_{x\to\infty} a(x) = A > 0, \quad \lim_{x\to\infty} b(x) = B > 0, \quad \lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty,$$

and

$$\frac{f'(x)}{g'(x)} + a(x)\frac{f(x)}{g(x)} = b(x).$$

Prove that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{B}{A+1}.$$

Solution. Let $0 < \varepsilon < A$ be an arbitrary real number. If x is sufficiently large then $f(x) > 0$, $g(x) > 0$, $|a(x) - A| < \varepsilon$, $|b(x) - B| < \varepsilon$ and

$$B - \varepsilon < b(x) = \frac{f'(x)}{g'(x)} + a(x)\frac{f(x)}{g(x)} < \frac{f'(x)}{g'(x)} + (A+\varepsilon)\frac{f(x)}{g(x)} < \tag{1}$$

$$< \frac{(A+\varepsilon)(A+1)}{A} \cdot \frac{f'(x)\big(g(x)\big)^A + A \cdot f(x) \cdot \big(g(x)\big)^{A-1} \cdot g'(x)}{(A+1) \cdot \big(g(x)\big)^A \cdot g'(x)} =$$

$$= \frac{(A+\varepsilon)(A+1)}{A} \cdot \frac{\Big(f(x) \cdot \big(g(x)\big)^A\Big)'}{\Big(\big(g(x)\big)^{A+1}\Big)'},$$

thus

$$\frac{\Big(f(x) \cdot \big(g(x)\big)^A\Big)'}{\Big(\big(g(x)\big)^{A+1}\Big)'} > \frac{A(B-\varepsilon)}{(A+\varepsilon)(A+1)}. \tag{2}$$

It can be similarly obtained that, for sufficiently large x,

$$\frac{\Big(f(x) \cdot \big(g(x)\big)^A\Big)'}{\Big(\big(g(x)\big)^{A+1}\Big)'} < \frac{A(B+\varepsilon)}{(A-\varepsilon)(A+1)}. \tag{3}$$

From $\varepsilon \to 0$, we have

$$\lim_{x\to\infty} \frac{\Big(f(x) \cdot \big(g(x)\big)^A\Big)'}{\Big(\big(g(x)\big)^{A+1}\Big)'} = \frac{B}{A+1}.$$

By l'Hospital's rule this implies

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f(x) \cdot \big(g(x)\big)^A}{\big(g(x)\big)^{A+1}} = \frac{B}{A+1}.$$