10^{th} **International Mathematical Competition for University Students**
Cluj-Napoca, July 2003

Day 1

1. (a) Let $a_1, a_2, \ldots$ be a sequence of real numbers such that $a_1 = 1$ and $a_{n+1} > \frac{3}{2}a_n$ for all n. Prove that the sequence

$$\frac{a_n}{\left(\frac{3}{2}\right)^{n-1}}$$

has a finite limit or tends to infinity. (10 points)

(b) Prove that for all $\alpha > 1$ there exists a sequence $a_1, a_2, \ldots$ with the same properties such that

$$\lim \frac{a_n}{\left(\frac{3}{2}\right)^{n-1}} = \alpha.$$

(10 points)

Solution. (a) Let $b_n = \dfrac{a_n}{\left(\frac{3}{2}\right)^{n-1}}$. Then $a_{n+1} > \frac{3}{2}a_n$ is equivalent to $b_{n+1} > b_n$, thus the sequence (b_n) is strictly increasing. Each increasing sequence has a finite limit or tends to infinity.

(b) For all $\alpha > 1$ there exists a sequence $1 = b_1 < b_2 < \ldots$ which converges to α. Choosing $a_n = \left(\frac{3}{2}\right)^{n-1} b_n$, we obtain the required sequence (a_n).

2. Let $a_1, a_2 \ldots, a_{51}$ be non-zero elements of a field. We simultaneously replace each element with the sum of the 50 remaining ones. In this way we get a sequence $b_1 \ldots, b_{51}$. If this new sequence is a permutation of the original one, what can be the characteristic of the field? (The characteristic of a field is p, if p is the smallest positive integer such that $\underbrace{x + x + \ldots + x}_{p} = 0$ for any element x of the field. If there exists no such p, the characteristic is 0.) (20 points)

Solution. Let $S = a_1 + a_2 + \cdots + a_{51}$. Then $b_1 + b_2 + \cdots + b_{51} = 50S$. Since $b_1, b_2, \cdots, b_{51}$ is a permutation of $a_1, a_2, \cdots, a_{51}$, we get $50S = S$, so $49S = 0$. Assume that the characteristic of the field is not equal to 7. Then $49S = 0$ implies that $S = 0$. Therefore $b_i = -a_i$ for $i = 1, 2, \cdots, 51$. On the other hand, $b_i = a_{\varphi(i)}$, where $\varphi \in S_{51}$. Therefore, if the characteristic is not 2, the sequence $a_1, a_2, \cdots, a_{51}$ can be partitioned into pairs $\{a_i, a_{\varphi(i)}\}$ of additive inverses. But this is impossible, since 51 is an odd number. It follows that the characteristic of the field is 7 or 2.

The characteristic can be either 2 or 7. For the case of 7, $x_1 = \ldots = x_{51} = 1$ is a possible choice. For the case of 2, any elements can be chosen such that $S = 0$, since then $b_i = -a_i = a_i$.

3. Let A be an $n \times n$ real matrix such that $3A^3 = A^2 + A + I$ (I is the identity matrix). Show that the sequence A^k converges to an idempotent matrix. (A matrix B is called idempotent if $B^2 = B$.) (20 points)

Solution. The minimal polynomial of A is a divisor of $3x^3 - x^2 - x - 1$. This polynomial has three different roots. This implies that A is diagonalizable: $A = C^{-1}DC$ where D is a diagonal matrix. The eigenvalues of the matrices A and D are all roots of polynomial $3x^3 - x^2 - x - 1$. One of the three roots is 1, the remaining two roots have smaller absolute value than 1. Hence, the diagonal elements of D^k, which are the kth powers of the eigenvalues, tend to either 0 or 1 and the limit $M = \lim D^k$ is idempotent. Then $\lim A^k = C^{-1}MC$ is idempotent as well.

4. Determine the set of all pairs (a, b) of positive integers for which the set of positive integers can be decomposed into two sets A and B such that $a \cdot A = b \cdot B$. (20 points)

Solution. Clearly a and b must be different since A and B are disjoint.

Let $\{a,b\}$ be a solution and consider the sets A, B such that $a\cdot A = b\cdot B$. Denoting $d=(a,b)$ the greatest common divisor of a and b, we have $a = d\cdot a_1,\ b = d\cdot b_1,\ (a_1,b_1)=1$ and $a_1\cdot A = b_1\cdot B$. Thus $\{a_1,b_1\}$ is a solution and it is enough to determine the solutions $\{a,b\}$ with $(a,b)=1$.

If $1\in A$ then $a\in a\cdot A = b\cdot B$, thus b must be a divisor of a. Similarly, if $1\in B$, then a is a divisor of b. Therefore, in all solutions, one of numbers a,b is a divisor of the other one.

Now we prove that if $n\ge 2$, then $(1,n)$ is a solution. For each positive integer k, let $f(k)$ be the largest non-negative integer for which $n^{f(k)}|k$. Then let $A=\{k:\ f(k)\text{ is odd}\}$ and $B=\{k:\ f(k)\text{ is even}\}$. This is a decomposition of all positive integers such that $A=n\cdot B$.

5. Let $g:\ [0,1]\to\mathbb{R}$ be a continuous function and let $f_n:\ [0,1]\to\mathbb{R}$ be a sequence of functions defined by $f_0(x)=g(x)$ and

$$f_{n+1}(x)=\frac{1}{x}\int_0^x f_n(t)dt \quad (x\in(0,1],\ n=0,1,2,\ldots).$$

Determine $\lim\limits_{n\to\infty} f_n(x)$ for every $x\in(0,1]$. (20 points)

B. We shall prove in two different ways that $\lim_{n\to\infty} f_n(x)=g(0)$ for every $x\in(0,1]$. (The second one is more lengthy but it tells us how to calculate f_n directly from g.)

Proof I. First we prove our claim for non-decreasing g. In this case, by induction, one can easily see that

1. each f_n is non-decrasing as well, and
2. $g(x)=f_0(x)\ge f_1(x)\ge f_2(x)\ge\ldots\ge g(0)\quad (x\in(0,1])$.

Then (2) implies that there exists

$$h(x)=\lim_{n\to\infty} f_n(x)\qquad (x\in(0,1]).$$

Clearly h is non-decreasing and $g(0)\le h(x)\le f_n(x)$ for any $x\in(0,1], n=0,1,2,\ldots$. Therefore to show that $h(x)=g(0)$ for any $x\in(0,1]$, it is enough to prove that $h(1)$ cannot be greater than $g(0)$.

Suppose that $h(1)>g(0)$. Then there exists a $0<\delta<1$ such that $h(1)>g(\delta)$. Using the definition, (2) and (1) we get

$$f_{n+1}(1)=\int_0^1 f_n(t)dt\le\int_0^\delta g(t)dt+\int_\delta^1 f_n(t)dt\le\delta g(\delta)+(1-\delta)f_n(1).$$

Hence

$$f_n(1)-f_{n+1}(1)\ge\delta(f_n(1)-g(\delta))\ge\delta(h(1)-g(\delta))>0,$$

so $f_n(1)\to-\infty$, which is a contradiction.

Similarly, we can prove our claim for non-increasing continuous functions as well.

Now suppose that g is an arbitrary continuous function on $[0,1]$. Let

$$M(x)=\sup_{t\in[0,x]} g(t),\qquad m(x)=\inf_{t\in[0,x]} g(t)\qquad\qquad (x\in[0,1])$$

Then on $[0,1]$ m is non-increasing, M is non-decreasing, both are continuous, $m(x)\le g(x)\le M(x)$ and $M(0)=m(0)=g(0)$. Define the sequences of functions $M_n(x)$ and $m_n(x)$ in the same way as f_n is defined but starting with $M_0=M$ and $m_0=m$.

Then one can easily see by induction that $m_n(x)\le f_n(x)\le M_n(x)$. By the first part of the proof, $\lim_n m_n(x)=m(0)=g(0)=M(0)=\lim_n M_n(x)$ for any $x\in(0,1]$. Therefore we must have $\lim_n f_n(x)=g(0)$.

Proof II. To make the notation clearer we shall denote the variable of f_j by x_j. By definition (and Fubini theorem) we get that

$$\begin{aligned} f_{n+1}(x_{n+1}) =& \frac{1}{x_{n+1}} \int_0^{x_{n+1}} \frac{1}{x_n} \int_0^{x_n} \frac{1}{x_{n-1}} \int_0^{x_{n-1}} \cdots \int_0^{x_2} \frac{1}{x_1} \int_0^{x_1} g(x_0) dx_0 dx_1 \dots dx_n \\ =& \frac{1}{x_{n+1}} \iint_{0 \le x_0 \le x_1 \le \dots \le x_n \le x_{n+1}} g(x_0) \frac{dx_0 dx_1 \dots dx_n}{x_1 \dots x_n} \\ =& \frac{1}{x_{n+1}} \int_0^{x_{n+1}} g(x_0) \left(\iint_{x_0 \le x_1 \le \dots \le x_n \le x_{n+1}} \frac{dx_1 \dots dx_n}{x_1 \dots x_n} \right) dx_0. \end{aligned}$$

Therefore with the notation

$$h_n(a,b) = \iint_{a \le x_1 \le \dots \le x_n \le b} \frac{dx_1 \dots dx_n}{x_1 \dots x_n}$$

and $x = x_{n+1}, t = x_0$ we have

$$f_{n+1}(x) = \frac{1}{x} \int_0^x g(t) h_n(t,x) dt.$$

Using that $h_n(a,b)$ is the same for any permutation of $x_1, \dots, x_n$ and the fact that the integral is 0 on any hyperplanes ($x_i = x_j$) we get that

$$\begin{aligned} n!\; h_n(a,b) &= \iint_{a \le x_1, \dots, x_n \le b} \frac{dx_1 \dots dx_n}{x_1 \dots x_n} = \int_a^b \cdots \int_a^b \frac{dx_1 \dots dx_n}{x_1 \dots x_n} \\ &= \left(\int_a^b \frac{dx}{x} \right)^n = (\log(b/a))^n. \end{aligned}$$

Therefore

$$f_{n+1}(x) = \frac{1}{x} \int_0^x g(t) \frac{(\log(x/t))^n}{n!} dt.$$

Note that if g is constant then the definition gives $f_n = g$. This implies on one hand that we must have

$$\frac{1}{x} \int_0^x \frac{(\log(x/t))^n}{n!} dt = 1$$

and on the other hand that, by replacing g by $g - g(0)$, we can suppose that $g(0) = 0$.

Let $x \in (0,1]$ and $\varepsilon > 0$ be fixed. By continuity there exists a $0 < \delta < x$ and an M such that $|g(t)| < \varepsilon$ on $[0, \delta]$ and $|g(t)| \le M$ on $[0,1]$. Since

$$\lim_{n \to \infty} \frac{(\log(x/\delta))^n}{n!} = 0$$

there exists an n_0 sucht that $\log(x/\delta))^n/n! < \varepsilon$ whenever $n \ge n_0$. Then, for any $n \ge n_0$, we have

$$\begin{aligned} |f_{n+1}(x)| &\le \frac{1}{x} \int_0^x |g(t)| \frac{(\log(x/t))^n}{n!} dt \\ &\le \frac{1}{x} \int_0^\delta \varepsilon \frac{(\log(x/t))^n}{n!} dt + \frac{1}{x} \int_\delta^x |g(t)| \frac{(\log(x/\delta))^n}{n!} dt \\ &\le \frac{1}{x} \int_0^x \varepsilon \frac{(\log(x/t))^n}{n!} dt + \frac{1}{x} \int_\delta^x M \varepsilon dt \\ &\le \varepsilon + M\varepsilon. \end{aligned}$$

Therefore $\lim_n f(x) = 0 = g(0)$.

6. Let $f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ be a polynomial with real coefficients. Prove that if all roots of f lie in the left half-plane $\{z \in \mathbb{C} : \text{ Re } z < 0\}$ then

$$a_k a_{k+3} < a_{k+1} a_{k+2}$$

holds for every $k = 0, 1, \ldots, n-3$. (20 points)

Solution. The polynomial f is a product of linear and quadratic factors, $f(z) = \prod_i (k_i z + l_i) \cdot \prod_j (p_j z^2 + q_j z + r_j)$, with $k_i, l_i, p_j, q_j, r_j \in \mathbb{R}$. Since all roots are in the left half-plane, for each i, k_i and l_i are of the same sign, and for each j, p_j, q_j, r_j are of the same sign, too. Hence, multiplying f by -1 if necessary, the roots of f don't change and f becomes the polynomial with all positive coefficients.

For the simplicity, we extend the sequence of coefficients by $a_{n+1} = a_{n+2} = \ldots = 0$ and $a_{-1} = a_{-2} = \ldots = 0$ and prove the same statement for $-1 \le k \le n-2$ by induction.

For $n \le 2$ the statement is obvious: a_{k+1} and a_{k+2} are positive and at least one of a_{k-1} and a_{k+3} is 0; hence, $a_{k+1} a_{k+2} > a_k a_{k+3} = 0$.

Now assume that $n \ge 3$ and the statement is true for all smaller values of n. Take a divisor of $f(z)$ which has the form $z^2 + pz + q$ where p and q are positive real numbers. (Such a divisor can be obtained from a conjugate pair of roots or two real roots.) Then we can write

$$f(z) = (z^2 + pz + q)(b_{n-2} z^{n-2} + \ldots + b_1 z + b_0) = (z^2 + pz + q) g(x). \tag{1}$$

The roots polynomial $g(z)$ are in the left half-plane, so we have $b_{k+1} b_{k+2} < b_k b_{k+3}$ for all $-1 \le k \le n-4$. Defining $b_{n-1} = b_n = \ldots = 0$ and $b_{-1} = b_{-2} = \ldots = 0$ as well, we also have $b_{k+1} b_{k+2} \le b_k b_{k+3}$ for all integer k.

Now we prove $a_{k+1} a_{k+2} > a_k a_{k+3}$. If $k = -1$ or $k = n-2$ then this is obvious since $a_{k+1} a_{k+2}$ is positive and $a_k a_{k+3} = 0$. Thus, assume $0 \le k \le n-3$. By an easy computation,

$$a_{k+1} a_{k+2} - a_k a_{k+3} =$$

$$= (qb_{k+1} + pb_k + b_{k-1})(qb_{k+2} + pb_{k+1} + b_k) - (qb_k + pb_{k-1} + b_{k-2})(qb_{k+3} + pb_{k+2} + b_{k+1}) =$$

$$= (b_{k-1} b_k - b_{k-2} b_{k+1}) + p(b_k^2 - b_{k-2} b_{k+2}) + q(b_{k-1} b_{k+2} - b_{k-2} b_{k+3}) +$$

$$+ p^2 (b_k b_{k+1} - b_{k-1} b_{k+2}) + q^2 (b_{k+1} b_{k+2} - b_k b_{k+3}) + pq(b_{k+1}^2 - b_{k-1} b_{k+3}).$$

We prove that all the six terms are non-negative and at least one is positive. Term $p^2(b_k b_{k+1} - b_{k-1} b_{k+2})$ is positive since $0 \le k \le n-3$. Also terms $b_{k-1} b_k - b_{k-2} b_{k+1}$ and $q^2(b_{k+1} b_{k+2} - b_k b_{k+3})$ are non-negative by the induction hypothesis.

To check the sign of $p(b_k^2 - b_{k-2} b_{k+2})$ consider

$$b_{k-1}(b_k^2 - b_{k-2} b_{k+2}) = b_{k-2}(b_k b_{k+1} - b_{k-1} b_{k+2}) + b_k (b_{k-1} b_k - b_{k-2} b_{k+1}) \ge 0.$$

If $b_{k-1} > 0$ we can divide by it to obtain $b_k^2 - b_{k-2} b_{k+2} \ge 0$. Otherwise, if $b_{k-1} = 0$, either $b_{k-2} = 0$ or $b_{k+2} = 0$ and thus $b_k^2 - b_{k-2} b_{k+2} = b_k^2 \ge 0$. Therefore, $p(b_k^2 - b_{k-2} b_{k+2}) \ge 0$ for all k. Similarly, $pq(b_{k+1}^2 - b_{k-1} b_{k+3}) \ge 0$.

The sign of $q(b_{k-1} b_{k+2} - b_{k-2} b_{k+3})$ can be checked in a similar way. Consider

$$b_{k+1}(b_{k-1} b_{k+2} - b_{k-2} b_{k+3}) = b_{k-1}(b_{k+1} b_{k+2} - b_k b_{k+3}) + b_{k+3}(b_{k-1} b_k - b_{k-2} b_{k+1}) \ge 0.$$

If $b_{k+1} > 0$, we can divide by it. Otherwise either $b_{k-2} = 0$ or $b_{k+3} = 0$. In all cases, we obtain $b_{k-1} b_{k+2} - b_{k-2} b_{k+3} \ge 0$.

Now the signs of all terms are checked and the proof is complete.