11$^{\text{th}}$ **International Mathematical Competition for University Students**
Skopje, 25–26 July 2004

Solutions for problems on Day 2

1. Let A be a real 4×2 matrix and B be a real 2×4 matrix such that

$$AB = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

Find BA. [20 points]

Solution. Let $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ and $B = \begin{pmatrix} B_1 & B_2 \end{pmatrix}$ where A_1, A_2, B_1, B_2 are 2×2 matrices. Then

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \begin{pmatrix} B_1 & B_2 \end{pmatrix} = \begin{pmatrix} A_1B_1 & A_1B_2 \\ A_2B_1 & A_2B_2 \end{pmatrix}$$

therefore, $A_1B_1 = A_2B_2 = I_2$ and $A_1B_2 = A_2B_1 = -I_2$. Then $B_1 = A_1^{-1}$, $B_2 = -A_1^{-1}$ and $A_2 = B_2^{-1} = -A_1$. Finally,

$$BA = \begin{pmatrix} B_1 & B_2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = B_1A_1 + B_2A_2 = 2I_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

2. Let $f, g\colon [a,b] \to [0,\infty)$ be continuous and non-decreasing functions such that for each $x \in [a,b]$ we have

$$\int_a^x \sqrt{f(t)}\,dt \le \int_a^x \sqrt{g(t)}\,dt$$

and $\int_a^b \sqrt{f(t)}\,dt = \int_a^b \sqrt{g(t)}\,dt$.

Prove that $\int_a^b \sqrt{1+f(t)}\,dt \ge \int_a^b \sqrt{1+g(t)}\,dt$. [20 points]

Solution. Let $F(x) = \int_a^x \sqrt{f(t)}\,dt$ and $G(x) = \int_a^x \sqrt{g(t)}\,dt$. The functions F, G are convex, $F(a) = 0 = G(a)$ and $F(b) = G(b)$ by the hypothesis. We are supposed to show that

$$\int_a^b \sqrt{1+\left(F'(t)\right)^2}\,dt \ge \int_a^b \sqrt{1+\left(G'(t)\right)^2}\,dt$$

i.e. The length ot the graph of F is $\ge$ the length of the graph of G. This is clear since both functions are convex, their graphs have common ends and the graph of F is *below* the graph of G — the length of the graph of F is the least upper bound of the lengths of the graphs of piecewise linear functions whose values at the points of non-differentiability coincide with the values of F, if a convex polygon P_1 is contained in a polygon P_2 then the perimeter of P_1 is $\le$ the perimeter of P_2.

3. Let D be the closed unit disk in the plane, and let $p_1, p_2, \dots, p_n$ be fixed points in D. Show that there exists a point p in D such that the sum of the distances of p to each of $p_1, p_2, \dots, p_n$ is greater than or equal to 1. [20 points]

Solution. considering as vectors, thoose p to be the unit vector which points into the opposite direction as $\sum\limits_{i=1}^n p_i$. Then, by the triangle inequality,

$$\sum_{i=1}^n |p - p_i| \ge \left| np - \sum_{i=1}^n p_i \right| = n + \left| \sum_{i=1}^n p_i \right| \ge n..$$

4. For $n \geq 1$ let M be an $n \times n$ complex matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, with multiplicities $m_1, m_2, \ldots, m_k$, respectively. Consider the linear operator L_M defined by $L_M(X) = MX + XM^T$, for any complex $n \times n$ matrix X. Find its eigenvalues and their multiplicities. (M^T denotes the transpose of M; that is, if $M = (m_{k,l})$, then $M^T = (m_{l,k})$.) [20 points]

Solution. We first solve the problem for the special case when the eigenvalues of M are distinct and all sums $\lambda_r + \lambda_s$ are different. Let λ_r and λ_s be two eigenvalues of M and $\vec{v}_r$, $\vec{v}_s$ eigenvectors associated to them, i.e. $M\vec{v}_j = \lambda \vec{v}_j$ for $j = r, s$. We have $M\vec{v}_r(\vec{v}_s)^T + \vec{v}_r(\vec{v}_s)^T M^T = (M\vec{v}_r)(\vec{v}_s)^T + \vec{v}_r\big(M\vec{v}_s\big)^T = \lambda_r \vec{v}_r(\vec{v}_s)^T + \lambda_s \vec{v}_r(\vec{v}_s)^T$, so $\vec{v}_r(\vec{v}_s)$ is an eigenmatrix of L_M with the eigenvalue $\lambda_r + \lambda_s$.

Notice that if $\lambda_r \neq \lambda_s$ then vectors $\vec{u}, \vec{w}$ are linearly independent and matrices $\vec{u}(\vec{w})^T$ and $\vec{w}(\vec{u})^T$ are linearly independent, too. This implies that the eigenvalue $\lambda_r + \lambda_s$ is double if $r \neq s$.

The map L_M maps n^2–dimensional linear space into itself, so it has at most n^2 eigenvalues. We already found n^2 eigenvalues, so there exists no more and the problem is solved for the special case.

In the general case, matrix M is a limit of matrices $M_1, M_2, \ldots$ such that each of them belongs to the special case above. By the continuity of the eigenvalues we obtain that the eigenvalues of L_M are

- $2\lambda_r$ with multiplicity m_r^2 $(r = 1, \ldots, k)$;
- $\lambda_r + \lambda_s$ with multiplicity $2m_r m_s$ $(1 \leq r < s \leq k)$.

(It can happen that the sums $\lambda_r + \lambda_s$ are not pairwise different; for those multiple values the multiplicities should be summed up.)

5. Prove that

$$\int_0^1 \int_0^1 \frac{dx\, dy}{x^{-1} + |\ln y| - 1} \leq 1. \quad \text{[20 points]}$$

Solution 1. First we use the inequality

$$x^{-1} - 1 \geq |\ln x|, \; x \in (0, 1],$$

which follows from

$$(x^{-1} - 1)\big|_{x=1} = |\ln x|\big|_{x=1} = 0,$$
$$(x^{-1} - 1)' = -\frac{1}{x^2} \leq -\frac{1}{x} = |\ln x|', \; x \in (0, 1].$$

Therefore

$$\int_0^1 \int_0^1 \frac{dx\, dy}{x^{-1} + |\ln y| - 1} \leq \int_0^1 \int_0^1 \frac{dx\, dy}{|\ln x| + |\ln y|} = \int_0^1 \int_0^1 \frac{dx\, dy}{|\ln(x \cdot y)|}.$$

Substituting $y = u/x$, we obtain

$$\int_0^1 \int_0^1 \frac{dx\, dy}{|\ln(x \cdot y)|} = \int_0^1 \left(\int_u^1 \frac{dx}{x} \right) \frac{du}{|\ln u|} = \int_0^1 |\ln u| \cdot \frac{du}{|\ln u|} = 1.$$

Solution 2. Substituting $s = x^{-1} - 1$ and $u = s - \ln y$,

$$\int_0^1 \int_0^1 \frac{dx\, dy}{x^{-1} + |\ln y| - 1} = \int_0^\infty \int_s^\infty \frac{e^{s-u}}{(s+1)^2 u} du ds = \int_0^\infty \left(\int_0^u \frac{e^s}{(s+1)^2} ds \right) \frac{e^{-u}}{u} ds du.$$

Since the function $\frac{e^s}{(s+1)^2}$ is convex,

$$\int_0^u \frac{e^s}{(s+1)^2} ds \leq \frac{u}{2} \left(\frac{e^u}{(u+1)^2} + 1 \right)$$

so

$$\int_0^1 \int_0^1 \frac{dx\, dy}{x^{-1} + |\ln y| - 1} \leq \int_0^\infty \frac{u}{2} \left(\frac{e^u}{(u+1)^2} + 1 \right) \frac{e^{-u}}{u} du = \frac{1}{2} \left(\int_0^\infty \frac{du}{(u+1)^2} + \int_0^\infty e^{-u} du \right) = 1.$$

6. For $n \geq 0$ define matrices A_n and B_n as follows: $A_0 = B_0 = (1)$ and for every $n > 0$

$$A_n = \begin{pmatrix} A_{n-1} & A_{n-1} \\ A_{n-1} & B_{n-1} \end{pmatrix} \text{ and } B_n = \begin{pmatrix} A_{n-1} & A_{n-1} \\ A_{n-1} & 0 \end{pmatrix}.$$

Denote the sum of all elements of a matrix M by $S(M)$. Prove that $S(A_n^{k-1}) = S(A_k^{n-1})$ for every $n, k \geq 1$. [20 points]

Solution. The quantity $S(A_n^{k-1})$ has a special combinatorical meaning. Consider an $n \times k$ table filled with 0's and 1's such that no 2×2 contains only 1's. Denote the number of such fillings by F_{nk}. The filling of each row of the table corresponds to some integer ranging from 0 to $2^n - 1$ written in base 2. F_{nk} equals to the number of k-tuples of integers such that every two consecutive integers correspond to the filling of $n \times 2$ table without 2×2 squares filled with 1's.

Consider binary expansions of integers i and j $\overline{i_n i_{n-1} \ldots i_1}$ and $\overline{j_n j_{n-1} \ldots j_1}$. There are two cases:

1. If $i_n j_n = 0$ then i and j can be consecutive iff $\overline{i_{n-1} \ldots i_1}$ and $\overline{j_{n-1} \ldots j_1}$ can be consequtive.
2. If $i_n = j_n = 1$ then i and j can be consecutive iff $i_{n-1} j_{n-1} = 0$ and $\overline{i_{n-2} \ldots i_1}$ and $\overline{j_{n-2} \ldots j_1}$ can be consecutive.

Hence i and j can be consecutive iff $(i+1, j+1)$-th entry of A_n is 1. Denoting this entry by $a_{i,j}$, the sum $S(A_n^{k-1}) = \sum_{i_1=0}^{2^n-1} \cdots \sum_{i_k=0}^{2^n-1} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k}$ counts the possible fillings. Therefore $F_{nk} = S(A_n^{k-1})$.

The the obvious statement $F_{nk} = F_{kn}$ completes the proof.