IMC 2012, Blagoevgrad, Bulgaria
Day 2, July 29, 2012

Problem 1. Consider a polynomial

$$f(x) = x^{2012} + a_{2011}\, x^{2011} + \ldots + a_1\, x + a_0 \,.$$

Albert Einstein and Homer Simpson are playing the following game. In turn, they choose one of the coefficients $a_0, \ldots, a_{2011}$ and assign a real value to it. Albert has the first move. Once a value is assigned to a coefficient, it cannot be changed any more. The game ends after all the coefficients have been assigned values.

Homer's goal is to make $f(x)$ divisible by a fixed polynomial $m(x)$ and Albert's goal is to prevent this.

(a) Which of the players has a winning strategy if $m(x) = x - 2012$?

(b) Which of the players has a winning strategy if $m(x) = x^2 + 1$?

(Proposed by Fedor Duzhin, Nanyang Technological University)

Solution. We show that Homer has a winning strategy in both part (a) and part (b).

(a) Notice that the last move is Homer's, and only the last move matters. Homer wins if and only if $f(2012) = 0$, i.e.

$$2012^{2012} + a_{2011}\, 2012^{2011} + \ldots + a_k\, 2012^k + \ldots + a_1\, 2012 + a_0 \ = 0. \tag{1}$$

Suppose that all of the coefficients except for a_k have been assigned values. Then Homer's goal is to establish (1) which is a linear equation on a_k. Clearly, it has a solution and hence Homer can win.

(b) Define the polynomials

$$g(y) = a_0 + a_2 y + a_4 y^2 + \ldots + a_{2010} y^{1005} + y^{1006} \quad \text{and} \quad h(y) = a_1 + a_3 y + a_5 y^2 + \ldots + a_{2011} y^{1005},$$

so $f(x) = g(x^2) + h(x^2) \cdot x$. Homer wins if he can achieve that $g(y)$ and $h(y)$ are divisible by $y+1$, i.e. $g(-1) = h(-1) = 0$.

Notice that both $g(y)$ and $h(y)$ have an even number of undetermined coefficients in the beginning of the game. A possible strategy for Homer is to follow Albert: whenever Albert assigns a value to a coefficient in g or h, in the next move Homer chooses the value for a coefficient in the same polynomial. This way Homer defines the last coefficient in g and he also chooses the last coefficient in h. Similarly to part (a), Homer can choose these two last coefficients in such a way that both $g(-1) = 0$ and $h(-1) = 0$ hold.

Problem 2. Define the sequence $a_0, a_1, \ldots$ inductively by $a_0 = 1$, $a_1 = \frac{1}{2}$ and

$$a_{n+1} = \frac{n a_n^2}{1 + (n+1) a_n} \quad \text{for } n \geq 1.$$

Show that the series $\displaystyle\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k}$ converges and determine its value.

(Proposed by Christophe Debry, KU Leuven, Belgium)

Solution. Observe that

$$ka_k = \frac{(1+(k+1)a_k)a_{k+1}}{a_k} = \frac{a_{k+1}}{a_k} + (k+1)a_{k+1} \quad \text{for all } k \geq 1,$$

and hence

$$\sum_{k=0}^{n} \frac{a_{k+1}}{a_k} = \frac{a_1}{a_0} + \sum_{k=1}^{n} (ka_k - (k+1)a_{k+1}) = \frac{1}{2} + 1 \cdot a_1 - (n+1)a_{n+1} = 1 - (n+1)a_{n+1} \tag{1}$$

for all $n \geq 0$.

By (1) we have $\sum\limits_{k=0}^{n} \frac{a_{k+1}}{a_k} < 1$. Since all terms are positive, this implies that the series $\sum\limits_{k=0}^{\infty} \frac{a_{k+1}}{a_k}$ is convergent. The sequence of terms, $\frac{a_{k+1}}{a_k}$ must converge to zero. In particular, there is an index n_0 such that $\frac{a_{k+1}}{a_k} < \frac{1}{2}$ for $n \geq n_0$. Then, by induction on n, we have $a_n < \frac{C}{2^n}$ with some positive constant C. From $na_n < \frac{Cn}{2^n} \to 0$ we get $na_n \to 0$, and therefore

$$\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{a_{k+1}}{a_k} = \lim_{n\to\infty} \Big(1 - (n+1)a_{n+1}\Big) = 1.$$

Remark. The inequality $a_n \leq \frac{1}{2^n}$ can be proved by a direct induction as well.

Problem 3. Is the set of positive integers n such that $n! + 1$ divides $(2012n)!$ finite or infinite?

(Proposed by Fedor Petrov, St. Petersburg State University)

Solution 1. Consider a positive integer n with $n! + 1 \big| (2012n)!$. It is well-known that for arbitrary nonnegative integers $a_1, \ldots, a_k$, the number $(a_1 + \ldots + a_k)!$ is divisible by $a_1! \cdot \ldots \cdot a_k!$. (The number of sequences consisting of a_1 digits 1, …, a_k digits k, is $\frac{(a_1+\ldots+a_k)!}{a_1!\cdot\ldots\cdot a_k!}$.) In particular, $(n!)^{2012}$ divides $(2012n)!$.

Since $n! + 1$ is co-prime with $(n!)^{2012}$, their product $(n! + 1)(n!)^{2012}$ also divides $(2012n)!$, and therefore

$$(n! + 1) \cdot (n!)^{2012} \leq (2012n)!\,.$$

By the known inequalities $\left(\frac{n+1}{e}\right)^n < n! \leq n^n$, we get

$$\left(\frac{n}{e}\right)^{2013n} < (n!)^{2013} < (n! + 1) \cdot (n!)^{2012} \leq (2012n)! < (2012n)^{2012n}$$
$$n < 2012^{2012} e^{2013}.$$

Therefore, there are only finitely many such integers n.

Remark. Instead of the estimate $\left(\frac{n+1}{e}\right)^n < n!$, we may apply the *Multinomial theorem*:

$$(x_1 + \cdots + x_\ell)^N = \sum_{k_1+\ldots+k_\ell=N} \frac{N!}{k_1! \cdot \ldots \cdot k_\ell!} x_1^{k_1} \ldots x_\ell^{k_\ell}.$$

Applying this to $N = 2012n$, $\ell = 2012$ and $x_1 = \ldots = x_\ell = 1$,

$$\frac{(2012n)!}{(n!)^{2012}} < (\underbrace{1 + 1 + \ldots + 1}_{2012})^{2012n} = 2012^{2012n},$$
$$n! < n! + 1 \leq \frac{(2012n)!}{(n!)^{2012}} < 2012^{2012n}.$$

On the right-hand side we have a geometric progression which increases slower than the factorial on the left-hand side, so this is true only for finitely many n.

Solution 2. Assume that $n > 2012$ is an integer with $n! + 1 \big| (2012n)!$. Notice that all prime divisors of $n! + 1$ are greater than n, and all prime divisors of $(2012n)!$ are smaller than $2012n$.

Consider a prime p with $n < p < 2012n$. Among $1, 2, \dots, 2012n$ there are $\left[\frac{2012n}{p}\right] < 2012$ numbers divisible by p; by $p^2 > n^2 > 2012n$, none of them is divisible by p^2. Therefore, the exponent of p in the prime factorization of $(2012n)!$ is at most 2011. Hence,

$$n! + 1 = \gcd\big(n! + 1, (2012n)!\big) < \prod_{n<p<2012p} p^{2011}.$$

Applying the inequality $\prod\limits_{p \le X} p < 4^X$,

$$n! < \prod_{n<p<2012p} p^{2011} < \left(\prod_{p<2012n} p\right)^{2011} < \left(4^{2012n}\right)^{2011} = \left(4^{2012\cdot 2011}\right)^n. \tag{2}$$

Again, we have a factorial on the left-and side and a geometric progression on the right-hand side.

Problem 4. Let $n \ge 2$ be an integer. Find all real numbers a such that there exist real numbers x_1, $\dots$, x_n satisfying

$$x_1(1 - x_2) = x_2(1 - x_3) = \ldots = x_{n-1}(1 - x_n) = x_n(1 - x_1) = a. \tag{1}$$

(Proposed by Walther Janous and Gerhard Kirchner, Innsbruck)

Solution. Throughout the solution we will use the notation $x_{n+1} = x_1$.

We prove that the set of possible values of a is

$$\left(-\infty, \frac{1}{4}\right] \bigcup \left\{ \frac{1}{4\cos^2 \frac{k\pi}{n}};\ k \in \mathbb{N},\ 1 \le k < \frac{n}{2} \right\}.$$

In the case $a \le \frac{1}{4}$ we can choose x_1 such that $x_1(1 - x_1) = a$ and set $x_1 = x_2 = \ldots = x_n$. Hence we will now suppose that $a > \frac{1}{4}$.

The system (1) gives the recurrence formula

$$x_{i+1} = \varphi(x_i) = 1 - \frac{a}{x_i} = \frac{x_i - a}{x_i}, \quad i = 1, \dots, n.$$

The fractional linear transform φ can be interpreted as a projective transform of the real projective line $\mathbb{R} \cup \{\infty\}$; the map φ is an element of the group $\mathrm{PGL}_2(\mathbb{R})$, represented by the linear transform $M = \begin{pmatrix} 1 & -a \\ 1 & 0 \end{pmatrix}$. (Note that $\det M \ne 0$ since $a \ne 0$.) The transform φ^n can be represented by M^n. A point $[u, v]$ (written in homogenous coordinates) is a fixed point of this transform if and only if $(u, v)^T$ is an eigenvector of M^n. Since the entries of M^n and the coordinates u, v are real, the corresponding eigenvalue is real, too.

The characteristic polynomial of M is $x^2 - x + a$, which has no real root for $a > \frac{1}{4}$. So M has two conjugate complex eigenvalues $\lambda_{1.2} = \frac{1}{2}\big(1 \pm \sqrt{4a - 1}i\big)$. The eigenvalues of M^n are $\lambda_{1,2}^n$, they are real if and only if $\arg \lambda_{1,2} = \pm \frac{k\pi}{n}$ with some integer k; this is equivalent with

$$\pm\sqrt{4a - 1} = \tan \frac{k\pi}{n},$$

$$a = \frac{1}{4}\left(1 + \tan^2 \tfrac{k\pi}{n}\right) = \frac{1}{4\cos^2 \frac{k\pi}{n}}.$$

If $\arg\lambda_1 = \frac{k\pi}{n}$ then $\lambda_1^n = \lambda_2^n$, so the eigenvalues of M^n are equal. The eigenvalues of M are distinct, so M and M^n have two linearly independent eigenvectors. Hence, M^n is a multiple of the identity. This means that the projective transform φ^n is the identity; starting from an arbitrary point $x_1 \in \mathbb{R}\cup\{\infty\}$, the cycle $x_1, x_2, \dots, x_n$ closes at $x_{n+1} = x_1$. There are only finitely many cycles $x_1, x_2, \dots, x_n$ containing the point ∞; all other cycles are solutions for (1).

Remark. If we write $x_j = P + Q\tan t_j$ where P, Q and $t_1, \dots, t_n$ are real numbers, the recurrence relation re-writes as

$$(P + Q\tan t_j)(1 - P - Q\tan t_{j+1}) = a$$
$$(1-P)Q\tan t_j - PQ\tan t_{j+1} = a + P(P-1) + Q^2\tan t_j \tan t_{j+1} \quad (j = 1, 2, \dots, n).$$

In view of the identity $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$, it is reasonable to choose $P = \frac{1}{2}$, and $Q = \sqrt{a - \frac{1}{4}}$. Then the recurrence leads to

$$t_j - t_{j+1} \equiv \arctan\sqrt{4a-1} \pmod{\pi}.$$

Problem 5. Let $c \geq 1$ be a real number. Let G be an abelian group and let $A \subset G$ be a finite set satisfying $|A + A| \leq c|A|$, where $X + Y := \{x + y \mid x \in X,\ y \in Y\}$ and $|Z|$ denotes the cardinality of Z. Prove that

$$|\underbrace{A + A + \ldots + A}_{k \text{ times}}| \leq c^k|A|$$

for every positive integer k. (Plünnecke's inequality)

(Proposed by Przemyslaw Mazur, Jagiellonian University)

Solution. Let B be a nonempty subset of A for which the value of the expression $c_1 = \frac{|A+B|}{|B|}$ is the least possible. Note that $c_1 \leq c$ since A is one of the possible choices of B.

Lemma 1. For any finite set $D \subset G$ we have $|A + B + D| \leq c_1|B + D|$.

Proof. Apply induction on the cardinality of D. For $|D| = 1$ the Lemma is true by the definition of c_1. Suppose it is true for some D and let $x \notin D$. Let $B_1 = \{y \in B \mid x + y \in B + D\}$. Then $B + (D \cup \{x\})$ decomposes into the union of two disjoint sets:

$$B + (D \cup \{x\}) = (B + D) \cup \big((B \setminus B_1) + \{x\}\big)$$

and therefore $|B + (D \cup \{x\})| = |B + D| + |B| - |B_1|$. Now we need to deal with the cardinality of the set $A + B + (D \cup \{x\})$. Writing $A + B + (D \cup \{x\}) = (A + B + D) \cup (A + B + \{x\})$ we count some of the elements twice: for example if $y \in B_1$, then $A + \{y\} + \{x\} \subset (A + B + D) \cap (A + B + \{x\})$. Therefore all the elements of the set $A + B_1 + \{x\}$ are counted twice and thus

$$|A + B + (D \cup \{x\})| \leq |A + B + D| + |A + B + \{x\}| - |A + B_1 + \{x\}| =$$
$$= |A + B + D| + |A + B| - |A + B_1| \leq c_1(|B + D| - |B| - |B_1|) = c_1|B + (D \cup \{x\})|,$$

where the last inequality follows from the inductive hypothesis and the observation that $\frac{|A+B|}{|B|} = c_1 \leq \frac{|A+B_1|}{|B_1|}$ (or B_1 is the empty set). □

Lemma 2. For every $k \geq 1$ we have $|\underbrace{A + \ldots + A}_{k \text{ times}} + B| \leq c_1^k|B|$.

Proof. Induction on k. For $k = 1$ the statement is true by definition of c_1. For greater k set $D = \underbrace{A + \ldots + A}_{k-1 \text{ times}}$ in the previous lemma: $|\underbrace{A + \ldots + A}_{k \text{ times}} + B| \leq c_1|\underbrace{A + \ldots + A}_{k-1 \text{ times}} + B| \leq c_1^k|B|$. □

Now notice that

$$|\underbrace{A + \ldots + A}_{k \text{ times}}| \leq |\underbrace{A + \ldots + A}_{k \text{ times}} + B| \leq c_1^k|B| \leq c^k|A|.$$

Remark. The proof above due to Giorgios Petridis and can be found at `http://gowers.wordpress.com/2011/02/10/a-new-way-of-proving-sumset-estimates/`