IMC 2013, Blagoevgrad, Bulgaria
Day 2, August 9, 2013

Problem 1. Let z be a complex number with $|z+1|>2$. Prove that $|z^3+1|>1$.

(Proposed by Walther Janous and Gerhard Kirchner, Innsbruck)

Solution. Since $z^3+1=(z+1)(z^2-z+1)$, it suffices to prove that $|z^2-z+1|\ge\frac{1}{2}$.

Assume that $z+1=re^{\varphi i}$, where $r=|z+1|>2$, and $\varphi=\arg(z+1)$ is some real number. Then

$$z^2-z+1=(re^{\varphi i}-1)^2-(re^{\varphi i}-1)+1=r^2e^{2\varphi i}-3re^{\varphi i}+3,$$

and

$$\begin{aligned}|z^2-z+1|^2&=\big(r^2e^{2\varphi i}-3re^{\varphi i}+3\big)\big(r^2e^{-2\varphi i}-3re^{-\varphi i}+3\big)=\\&=r^4+9r^2+9-(6r^3+18r)\cos\varphi+6r^2\cos 2\varphi=\\&=r^4+9r^2+9-(6r^3+18r)\cos\varphi+6r^2(2\cos^2\varphi-1)=\\&=12\left(r\cos\varphi-\frac{r^2+3}{4}\right)^2+\frac{1}{4}(r^2-3)^2>0+\frac{1}{4}=\frac{1}{4}.\end{aligned}$$

This finishes the proof.

Problem 2. Let p and q be relatively prime positive integers. Prove that

$$\sum_{k=0}^{pq-1}(-1)^{\left\lfloor\frac{k}{p}\right\rfloor+\left\lfloor\frac{k}{q}\right\rfloor}=\begin{cases}0 & \text{if } pq \text{ is even,}\\ 1 & \text{if } pq \text{ is odd.}\end{cases}\qquad(*)$$

(Here $\lfloor x\rfloor$ denotes the integer part of x.)

(Proposed by Alexander Bolbot, State University, Novosibirsk)

Solution. Suppose first that pq is even (which implies that p and q have opposite parities), and let $a_k=(-1)^{\left\lfloor\frac{k}{p}\right\rfloor+\left\lfloor\frac{k}{q}\right\rfloor}$. We show that $a_k+a_{pq-1-k}=0$, so the terms on the left-and side of $(*)$ cancel out in pairs.

For every positive integer k we have $\left\{\frac{k}{p}\right\}+\left\{\frac{p-1-k}{p}\right\}=\frac{p-1}{p}$, hence

$$\left\lfloor\frac{k}{p}\right\rfloor+\left\lfloor\frac{pq-1-k}{p}\right\rfloor=\left(\frac{k}{p}-\left\{\frac{k}{p}\right\}\right)+\left(\frac{pq-1-k}{p}-\left\{\frac{-1-k}{p}\right\}\right)=\frac{pq-1}{p}-\frac{p-1}{p}=q-1$$

and similarly

$$\left\lfloor\frac{pq-1-k}{q}\right\rfloor+\left\lfloor\frac{k}{q}\right\rfloor=p-1.$$

Since p and q have opposite parities, it follows that $\left\lfloor\frac{k}{p}\right\rfloor+\left\lfloor\frac{k}{q}\right\rfloor$ and $\left\lfloor\frac{pq-1-k}{p}\right\rfloor+\left\lfloor\frac{pq-1-k}{q}\right\rfloor$ have opposite parities and therefore $a_{pq-1-k}=-a_k$.

Now suppose that pq is odd. For every index k, denote by p_k and q_k the remainders of k modulo p and q, respectively. (I.e., $0\le p_k<p$ and $0\le q_k<q$ such that $k\equiv p_k \pmod p$ and $k\equiv q_k \pmod q$.)

Notice that

$$\left\lfloor\frac{k}{p}\right\rfloor+\left\lfloor\frac{k}{q}\right\rfloor\equiv p\left\lfloor\frac{k}{p}\right\rfloor+q\left\lfloor\frac{k}{q}\right\rfloor=(k-p_k)+(k-q_k)\equiv p_k+q_k \pmod 2.$$

Since p and q are co-prime, by the Chinese remainder theorem the map $k \mapsto (p_k, q_k)$ is a bijection between the sets $\{0, 1, \ldots, pq-1\}$ and $\{0, 1, \ldots, p-1\} \times \{0, 1, \ldots, q-1\}$. Hence

$$\sum_{k=0}^{pq-1} (-1)^{\lfloor \frac{k}{p} \rfloor + \lfloor \frac{k}{q} \rfloor} = \sum_{k=0}^{pq-1} (-1)^{p_k+q_k} = \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} (-1)^{i+j} = \left(\sum_{i=0}^{p-1} (-1)^i \right) \cdot \left(\sum_{j=0}^{q-1} (-1)^j \right) = 1.$$

Problem 3. Suppose that $v_1, \ldots, v_d$ are unit vectors in $\mathbb{R}^d$. Prove that there exists a unit vector u such that

$$|u \cdot v_i| \leq 1/\sqrt{d}$$

for $i = 1, 2, \ldots, d$.

(Here $\cdot$ denotes the usual scalar product on $\mathbb{R}^d$.)

(Proposed by Tomasz Tkocz, University of Warwick)

Solution. If $v_1, \ldots, v_d$ are linearly dependent then we can simply take a unit vector u perpendicular to $\text{span}(v_1, \ldots, v_d)$. So assume that $v_1, \ldots, v_d$ are linearly independent. Let $w_1, \ldots, w_d$ be the dual basis of $(v_1, \ldots, v_d)$, i.e.

$$w_i \cdot v_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{for } 1 \leq i, j \leq d.$$

From $w_i \cdot v_i = 1$ we have $|w_i| \geq 1$.

For every sequence $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \{+1, -1\}^d$ of signs define $u_\varepsilon = \sum_{i=1}^d \varepsilon_i w_i$. Then we have

$$|u_\varepsilon \cdot v_k| = \left| \sum_{i=1}^d \varepsilon_i (w_i \cdot v_k) \right| = \left| \sum_{i=1}^d \varepsilon_i \delta_{ik} \right| = |\varepsilon_k| = 1 \quad \text{for } k = 1, \ldots, d.$$

Now estimate the average of $|u_\varepsilon|^2$.

$$\frac{1}{2^d} \sum_{\varepsilon \in \{+1,-1\}^n} |u_\varepsilon|^2 = \frac{1}{2^d} \sum_{\varepsilon \in \{+1,-1\}^n} \left(\sum_{i=1}^d \varepsilon_i w_i \right) \cdot \left(\sum_{j=1}^d \varepsilon_j w_j \right) =$$

$$= \sum_{i=1}^d \sum_{j=1}^d (w_i \cdot w_j) \left(\frac{1}{2^d} \sum_{\varepsilon \in \{+1,-1\}^n} \varepsilon_i \varepsilon_j \right) = \sum_{i=1}^d \sum_{j=1}^d (w_i \cdot w_j) \delta_{ij} = \sum_{i=1}^d |w_i|^2 \geq d.$$

It follows that there is a ε such that $|u_\varepsilon|^2 \geq d$. For that ε the vector $u = \dfrac{u_\varepsilon}{|u_\varepsilon|}$ satisfies the conditions.

Problem 4. Does there exist an infinite set M consisting of positive integers such that for any $a, b \in M$, with $a < b$, the sum $a + b$ is square-free?

(A positive integer is called square-free if no perfect square greater than 1 divides it.)

(Proposed by Fedor Petrov, St. Petersburg State University)

Solution. The answer is yes. We construct an infinite sequence $1 = n_1 < 2 = n_2 < n_3 < \ldots$ so that $n_i + n_j$ is square-free for all $i < j$. Suppose that we already have some numbers $n_1 < \ldots < n_k$ $(k \geq 2)$, which satisfy this condition and find a suitable number n_{k+1} to be the next element of the sequence.

We will choose n_{k+1} of the form $n_{k+1} = 1 + Mx$, with $M = ((n_1 + \ldots + n_k + 2k)!)^2$ and some positive integer x. For $i = 1, 2, \ldots, k$ we have $n_i + n_{k+1} = 1 + Mx + n_i = (1 + n_i)m_i$, where m_i and M are co-prime, so any perfect square dividing $1 + Mx + n_i$ is co-prime with M.

In order to find a suitable x, take a large N and consider the values $x = 1, 2, \dots, N$. If a value $1 \le x \le N$ is not suitable, this means that there is an index $1 \le i \le k$ and some prime p such that $p^2 | 1 + Mx + n_i$. For $p \le 2k$ this is impossible because $p|M$. Moreover, we also have $p^2 \le 1 + Mx + n_i < M(N+1)$, so $2k < p < \sqrt{M(N+1)}$.

For any fixed i and p, the values for x for which $p^2 | 1 + Mx + n_i$ form an arithmetic progression with difference p^2. Therefore, there are at most $\frac{N}{p^2} + 1$ such values. In total, the number of unsuitable values x is less than

$$\sum_{i=1}^{k} \sum_{2k<p<\sqrt{M(N+1)}} \left(\frac{N}{p^2} + 1\right) < k \cdot \left(N \sum_{p>2k} \frac{1}{p^2} + \sum_{p<\sqrt{M(N+1)}} 1 \right) <$$

$$< kN \sum_{p>2k} \left(\frac{1}{p-1} - \frac{1}{p}\right) + k\sqrt{M(N+1)} < \frac{N}{2} + k\sqrt{M(N+1)}.$$

If N is big enough then this is less than N, and there exist a suitable choice for x.

Problem 5. Consider a circular necklace with 2013 beads. Each bead can be painted either white or green. A painting of the necklace is called *good*, if among any 21 successive beads there is at least one green bead. Prove that the number of good paintings of the necklace is odd.

(Two paintings that differ on some beads, but can be obtained from each other by rotating or flipping the necklace, are counted as different paintings.)

(Proposed by Vsevolod Bykov and Oleksandr Rybak, Kiev)

Solution 1. For $k = 0, 1, \dots$ denote by N_k be the number of *good open laces*, consisting of k (white and green) beads in a row, such that among any 21 successive beads there is at least one green bead. For $k \le 21$ all laces have this property, so $N_k = 2^k$ for $0 \le k \le 20$; in particular, N_0 is odd, and $N_1, \dots, N_{20}$ are even.

For $k \ge 21$, there must be a green bead among the last 21 ones. Suppose that the last green bead is at the ℓth position; then $\ell \ge k - 20$. The previous $\ell - 1$ beads have $N_{\ell-1}$ good colorings, and every such good coloring provides a good lace of length k. Hence,

$$N_k = N_{k-1} + N_{k-2} + \dots + N_{k-21} \quad \text{for } k \ge 21. \tag{1}$$

From (1) we can see that $N_{21} = N_0 + \dots + N_{20}$ is odd, and $N_{22} = N_1 + \dots + N_{21}$ is also odd.

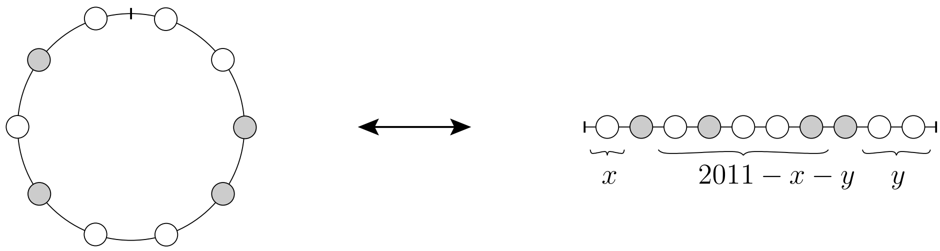
Applying (1) again to the term N_{k-1},

$$N_k = N_{k-1} + \dots + N_{k-21} = \Big(N_{k-2} + \dots + N_{k-22}\Big) + N_{k-2} + \dots + N_{k-21} \equiv N_{k-22} \pmod 2$$

so the sequence of parities in (N_k) is periodic with period 22. We conclude that

- N_k is odd if $k \equiv 0 \pmod{22}$ or $k \equiv 21 \pmod{22}$;
- N_k is even otherwise.

Now consider the good circular necklaces of 2013 beads. At a fixed point between two beads cut each. The resulting open lace may have some consecutive white beads at the two ends, altogether at most 20. Suppose that there are x white beads at the beginning and y white beads at the end; then we have $x, y \ge$ and $x + y \le 20$, and we have a good open lace in the middle, between the first and the last green beads. That middle lace consist of $2011 - x - y$ beads. So, for any fixed values of x and y the number of such cases is $N_{2011-x-y}$.



It is easy to see that from such a good open lace we can reconstruct the original circular lace. Therefore, the number of good circular necklaces is

$$\sum_{x+y\le 20} N_{2011-x-y} = N_{2011}+2N_{2010}+3N_{2009}+\ldots+21N_{1991} \equiv N_{2011}+N_{2009}+N_{2007}+\ldots+N_{1991} \pmod 2.$$

By $91\cdot 22-1=2001$ the term N_{2001} is odd, the other terms are all even, so the number of the good circular necklaces is odd.

Solution 2 (by Yoav Krauz, Israel). There is just one good monochromatic necklace. Let us count the parity of good necklaces having both colors.

For each necklace, we define an *adjusted necklace*, so that at position 0 we have a white bead and at position 1 we have a green bead. If the necklace is satisfying the condition, it corresponds to itself; if both beads 0 and 1 are white we rotate it (so that the bead 1 goes to place 0) until bead 1 becomes green; if bead 1 is green, we rotate it in the opposite direction until the bead 0 will be white. This procedure is called adjusting, and the place between the white and green bead which are rotated into places 0 and 1 will be called *distinguished place*. The interval consisting of the subsequent green beads after the distinguished place and subsequent white beads before it will be called *distinguished interval*.

For each adjusted necklace we have several necklaces corresponding to it, and the number of them is equal to the length of distinguished interval (the total number of beads in it) minus 1. Since we count only the parity, we can disregard the adjusted necklaces with even distinguished intervals and count once each adjusted necklace with odd distinguished interval.

Now we shall prove that the number of necklaces with odd distinguished intervals is even by grouping them in pairs. The pairing is the following. If the number of white beads with in the distinguished interval is even, we turn the last white bead (at the distinguished place) into green. The white interval remains, since a positive even number minus 1 is still positive. If the number of white beads in the distinguished interval is odd, we turn the green bead next to the distinguished place into white. The green interval remains since it was even; the white interval was odd and at most 19 so it will become even and at most 20, so we still get a good necklace.

This pairing on good necklaces with distinguished intervals of odd length shows, that the number of such necklaces is even; hence the total number of all good necklaces using both colors is even. Therefore, together with monochromatic green necklace, the number of good necklaces is odd.