IMC 2015, Blagoevgrad, Bulgaria

Day 2, July 30, 2015

Problem 6. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,(n+1)} < 2.$$

(Proposed by Ivan Krijan, University of Zagreb)

Solution. We prove that

$$\frac{1}{\sqrt{n}\,(n+1)} < \frac{2}{\sqrt{n}} - \frac{2}{\sqrt{n+1}}. \tag{1}$$

Multiplying by $\sqrt{n}(n+1)$, the inequality (1) is equivalent with

$$1 < 2(n+1) - 2\sqrt{n(n+1)}$$

$$2\sqrt{n(n+1)} < n + (n+1)$$

which is true by the AM-GM inequality.

Applying (1) to the terms in the left-hand side,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,(n+1)} < \sum_{n=1}^{\infty} \left(\frac{2}{\sqrt{n}} - \frac{2}{\sqrt{n+1}} \right) = 2.$$

Problem 7. Compute

$$\lim_{A\to+\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} \mathrm{d}x\,.$$

(Proposed by Jan Šustek, University of Ostrava)

Solution 1. We prove that

$$\lim_{A\to+\infty} \frac{1}{A} \int_1^A A^{\frac{1}{x}} \mathrm{d}x = 1\,.$$

For $A > 1$ the integrand is greater than 1, so

$$\frac{1}{A} \int_1^A A^{\frac{1}{x}} \mathrm{d}x > \frac{1}{A} \int_1^A 1\mathrm{d}x = \frac{1}{A}(A-1) = 1 - \frac{1}{A}.$$

In order to find a tight upper bound, fix two real numbers, $\delta > 0$ and $K > 0$, and split the interval into three parts at the points $1+\delta$ and $K \log A$. Notice that for sufficiently large A (i.e., for $A > A_0(\delta, K)$ with some $A_0(\delta, K) > 1$) we have $1 + \delta < K \log A < A$.) For $A > 1$ the integrand is decreasing, so we can estimate it by its value at the starting points of the intervals:

$$\frac{1}{A} \int_1^A A^{\frac{1}{x}} \mathrm{d}x = \frac{1}{A} \left(\int_1^{1+\delta} + \int_{1+\delta}^{K\log A} + \int_{K\log A}^{A} \right) <$$

$$= \frac{1}{A} \left(\delta \cdot A + (K \log A - 1 - \delta) A^{\frac{1}{1+\delta}} + (A - K\log A) A^{\frac{1}{K \log A}} \right) <$$

$$< \frac{1}{A} \left(\delta A + K A^{\frac{1}{1+\delta}} \log A + A \cdot A^{\frac{1}{K\log A}} \right) = \delta + K A^{-\frac{\delta}{1+\delta}} \log A + e^{\frac{1}{K}}.$$

Hence, for $A > A_0(\delta, K)$ we have

$$1 - \frac{1}{A} < \frac{1}{A} \int_1^A A^{\frac{1}{x}} \mathrm{d}x < \delta + K A^{-\frac{\delta}{1+\delta}} \log A + e^{\frac{1}{K}}.$$

Taking the limit $A \to \infty$ we obtain

$$1 \le \liminf_{A\to\infty} \frac{1}{A}\int_1^A A^{\frac{1}{x}}\mathrm{d}x \le \limsup_{A\to\infty} \frac{1}{A}\int_1^A A^{\frac{1}{x}}\mathrm{d}x \le \delta + e^{\frac{1}{K}}.$$

Now from $\delta \to +0$ and $K \to \infty$ we get

$$1 \le \liminf_{A\to\infty} \frac{1}{A}\int_1^A A^{\frac{1}{x}}\mathrm{d}x \le \limsup_{A\to\infty} \frac{1}{A}\int_1^A A^{\frac{1}{x}}\mathrm{d}x \le 1,$$

so $\liminf_{A\to\infty} \frac{1}{A}\int_1^A A^{\frac{1}{x}}\mathrm{d}x = \limsup_{A\to\infty} \frac{1}{A}\int_1^A A^{\frac{1}{x}}\mathrm{d}x = 1$ and therefore

$$\lim_{A\to+\infty} \frac{1}{A}\int_1^A A^{\frac{1}{x}}\mathrm{d}x = 1\,.$$

Solution 2. We will employ l'Hospital's rule.

Let $f(A,x) = A^{\frac{1}{x}}$, $g(A,x) = \frac{1}{x}A^{\frac{1}{x}}$, $F(A) = \int_1^A f(A,x)\mathrm{d}x$ and $G(A) = \int_1^A g(A,x)\mathrm{d}x$. Since $\frac{\partial}{\partial A}f$ and $\frac{\partial}{\partial A}g$ are continuous, the parametric integrals $F(A)$ and $G(A)$ are differentiable with respect to A, and

$$F'(A) = f(A,A) + \int_1^A \frac{\partial}{\partial A} f(A,x)\mathrm{d}x = A^{\frac{1}{A}} + \int_1^A \frac{1}{x}A^{\frac{1}{x}-1}\mathrm{d}x = A^{\frac{1}{A}} + \frac{1}{A}G(A),$$

and

$$G'(A) = g(A,A) + \int_1^A \frac{\partial}{\partial A} g(A,x)\mathrm{d}x = \frac{A^{\frac{1}{A}}}{A} + \int_1^A \frac{1}{x^2}A^{\frac{1}{x}-1}\mathrm{d}x =$$

$$= A^{\frac{1}{A}} + \left[\frac{-1}{\log A}A^{\frac{1}{x}-1}\right]_1^A = \frac{A^{\frac{1}{A}}}{A} - \frac{A^{\frac{1}{A}}}{A\log A} + \frac{1}{\log A}.$$

Since $\lim_{A\to\infty} A^{\frac{1}{A}} = 1$, we can see that $\lim_{A\to\infty} G'(A) = 0$. Aplying l'Hospital's rule to $\lim_{A\to\infty} \frac{G(A)}{A}$ we get

$$\lim_{A\to\infty} \frac{G(A)}{A} = \lim_{A\to\infty} \frac{G'(A)}{1} = 0,$$

so

$$\lim_{A\to\infty} F'(A) = \lim_{A\to\infty}\left(A^{\frac{1}{A}} + \frac{G(A)}{A}\right) = 1 + 0 = 1.$$

Now applying l'Hospital's rule to $\lim_{A\to\infty} \frac{F(A)}{A}$ we get

$$\lim_{A\to+\infty} \frac{1}{A}\int_1^A A^{\frac{1}{x}}\mathrm{d}x = \lim_{A\to\infty} \frac{F(A)}{A} = \lim_{A\to\infty} \frac{F'(A)}{1} = 1.$$

Problem 8. Consider all 26^{26} words of length 26 in the Latin alphabet. Define the *weight* of a word as $1/(k+1)$, where k is the number of letters not used in this word. Prove that the sum of the weights of all words is 3^{75}.

(Proposed by Fedor Petrov, St. Petersburg State University)

Solution. Let $n = 26$, then $3^{75} = (n+1)^{n-1}$. We use the following well-known

Lemma. If $f(x)$ is a polynomial of degree at most n, then its $(n+1)$-st finite difference vanishes: $\Delta^{n+1} f(x) := \sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}f(x+i) \equiv 0$.

Proof. If Δ is the operator which maps $f(x)$ to $f(x+1) - f(x)$, then Δ^{n+1} is indeed $(n+1)$-st power of Δ and the claim follows from the observation that Δ decreases the power of a polynomial.

In other words, $f(x) = \sum_{i=1}^{n+1}(-1)^{i+1}\binom{n+1}{i}f(x+i)$. Applying this for $f(x) = (n-x)^n$, substituting $x = -1$ and denoting $i = j+1$ we get

$$(n+1)^n = \sum_{j=0}^{n}(-1)^j\binom{n+1}{j+1}(n-j)^n = (n+1)\sum_{j=0}^{n}\binom{n}{j}\cdot\frac{(-1)^j}{j+1}\cdot(n-j)^n.$$

The j-th summand $\binom{n}{j} \cdot \frac{(-1)^j}{j+1} \cdot (n-j)^n$ may be interpreted as follows: choose j letters, consider all $(n-j)^n$ words without those letters and sum up $\frac{(-1)^j}{j+1}$ over all those words. Now we change the order of summation, counting at first by words. For any fixed word W with k absent letters we get $\sum_{j=0}^{k} \binom{k}{j} \cdot \frac{(-1)^j}{j+1} = \frac{1}{k+1} \cdot \sum_{j=0}^{k} (-1)^j \cdot \binom{k+1}{j+1} = \frac{1}{k+1}$, since the alternating sum of binomial coefficients $\sum_{j=-1}^{k} (-1)^j \cdot \binom{k+1}{j+1}$ vanishes. That is, after changing order of summation we get exactly initial sum, and it equals $(n+1)^{n-1}$.

Problem 9. An $n \times n$ complex matrix A is called *t-normal* if $AA^t = A^t A$ where A^t is the transpose of A. For each n, determine the maximum dimension of a linear space of complex $n \times n$ matrices consisting of t-normal matrices.

(Proposed by Shachar Carmeli, Weizmann Institute of Science)

Solution.

Answer: The maximum dimension of such a space is $\frac{n(n+1)}{2}$.

The number $\frac{n(n+1)}{2}$ can be achieved, for example the symmetric matrices are obviously t-normal and they form a linear space with dimension $\frac{n(n+1)}{2}$. We shall show that this is the maximal possible dimension.

Let M_n denote the space of $n \times n$ complex matrices, let $S_n \subset M_n$ be the subspace of all symmetric matrices and let $A_n \subset M_n$ be the subspace of all anti-symmetric matrices, i.e. matrices A for which $A^t = -A$.

Let $V \subset M_n$ be a linear subspace consisting of t-normal matrices. We have to show that $\dim(V) \le \dim(S_n)$. Let $\pi : V \to S_n$ denote the linear map $\pi(A) = A + A^t$. We have

$$\dim(V) = \dim(\mathrm{Ker}\,(\pi)) + \dim(\mathrm{Im}\,(\pi))$$

so we have to prove that $\dim(\mathrm{Ker}\,(\pi)) + \dim(\mathrm{Im}\,(\pi)) \le \dim(S_n)$. Notice that $\mathrm{Ker}\,(\pi) \subseteq A_n$.

We claim that for every $A \in \mathrm{Ker}\,(\pi)$ and $B \in V$, $A\pi(B) = \pi(B)A$. In other words, $\mathrm{Ker}\,(\pi)$ and $\mathrm{Im}\,(\pi)$ commute. Indeed, if $A, B \in V$ and $A = -A^t$ then

$$\begin{gathered}(A+B)(A+B)^t = (A+B)^t(A+B) \Leftrightarrow \\ \Leftrightarrow AA^t + AB^t + BA^t + BB^t = A^tA + A^tB + B^tA + B^tB \Leftrightarrow \\ \Leftrightarrow AB^t - BA = -AB + B^tA \Leftrightarrow A(B+B^t) = (B+B^t)A \Leftrightarrow \\ \Leftrightarrow A\pi(B) = \pi(B)A.\end{gathered}$$

Our bound on the dimension on V follows from the following lemma:

Lemma. Let $X \subseteq S_n$ and $Y \subseteq A_n$ be linear subspaces such that every element of X commutes with every element of Y. Then

$$\dim(X) + \dim(Y) \le \dim(S_n)$$

Proof. Without loss of generality we may assume $X = Z_{S_n}(Y) := \{x \in S_n : xy = yx \quad \forall y \in Y\}$. Define the bilinear map $B : S_n \times A_n \to \mathbb{C}$ by $B(x,y) = \mathrm{tr(d[x,y])}$ where $[x,y] = xy - yx$ and $d = diag(1, ..., n)$ is the matrix with diagonal elements $1, ..., n$ and zeros off the diagonal. Clearly $B(X,Y) = \{0\}$. Furthermore, if $y \in Y$ satisfies that $B(x,y) = 0$ for all $x \in S_n$ then $\mathrm{tr(d[x,y])} = -\mathrm{tr([[d,x],y])} = 0$ for every $x \in S_n$.

We claim that $\{[d,x] : x \in S_n\} = A_n$. Let E_i^j denote the matrix with 1 in the entry (i,j) and 0 in all other entries. Then a direct computation shows that $[d, E_i^j] = (j-i)E_i^j$ and therefore $[d, E_i^j + E_j^i] = (j-i)(E_i^j - E_j^i)$ and the collection $\{(j-i)(E_i^j - E_j^i)\}_{1 \le i < j \le n}$ span A_n for $i \ne j$.

It follows that if $B(x,y) = 0$ for all $x \in S_n$ then $\mathrm{tr(yz)} = 0$ for every $z \in A_n$. But then, taking $z = \bar{y}$, where $\bar{y}$ is the entry-wise complex conjugate of y, we get $0 = \mathrm{tr(y\bar{y})} = -\mathrm{tr(y\bar{y}^t)}$ which is the sum of squares of all the entries of y. This means that $y = 0$.

It follows that if $y_1, ..., y_k \in Y$ are linearly independent then the equations

$$B(x, y_1) = 0, \quad \ldots, \quad B(x, y_k) = 0$$

are linearly independent as linear equations in x, otherwise there are $a_1, ..., a_k$ such that $B(x, a_1 y_1 + ... + a_k y_k) = 0$ for every $x \in S_n$, a contradiction to the observation above. Since the solution of k linearly independent linear equations is of codimension k,

$$\dim(\{x \in S_n : [x, y_i] = 0, \text{ for } i = 1, .., k\}) \leq$$

$$\leq \dim(x \in S_n : B(x, y_i) = 0 \text{ for } i = 1, ..., k) = \dim(S_n) - k.$$

The lemma follows by taking $y_1, ..., y_k$ to be a basis of Y.

Since $\mathrm{Ker}\,(\pi)$ and $\mathrm{Im}\,(\pi)$ commute, by the lemma we deduce that

$$\dim(V) = \dim(\mathrm{Ker}\,(\pi)) + \dim(\mathrm{Im}\,(\pi)) \leq \dim(S_n) = \frac{n(n+1)}{2}.$$

Problem 10. Let n be a positive integer, and let $p(x)$ be a polynomial of degree n with integer coefficients. Prove that

$$\max_{0 \leq x \leq 1} \left|p(x)\right| > \frac{1}{e^n}.$$

(Proposed by Géza Kós, Eötvös University, Budapest)

Solution. Let

$$M = \max_{0 \leq x \leq 1} \left|p(x)\right|.$$

For every positive integer k, let

$$J_k = \int_0^1 \big(p(x)\big)^{2k} \mathrm{d}x.$$

Obviously $0 < J_k < M^{2k}$ is a rational number. If $(p(x))^{2k} = \sum\limits_{i=0}^{2kn} a_{k,i} x^i$ then $J_k = \sum\limits_{i=0}^{2kn} \frac{a_{k,i}}{i+1}$. Taking the least common denominator, we can see that $J_k \geq \dfrac{1}{\mathrm{lcm}(1, 2, \ldots, 2kn+1)}$.

An equivalent form of the prime number theorem is that $\log \mathrm{lcm}(1, 2, \ldots, N) \sim N$ if $N \to \infty$. Therefore, for every $\varepsilon > 0$ and sufficiently large k we have

$$\mathrm{lcm}(1, 2, \ldots, 2kn+1) < e^{(1+\varepsilon)(2kn+1)}$$

and therefore

$$M^{2k} > J_k \geq \frac{1}{\mathrm{lcm}(1, 2, \ldots, 2kn+1)} > \frac{1}{e^{(1+\varepsilon)(2kn+1)}},$$

$$M > \frac{1}{e^{(1+\varepsilon)(n+\frac{1}{2k})}}.$$

Taking $k \to \infty$ and then $\varepsilon \to +0$ we get

$$M \geq \frac{1}{e^n}.$$

Since e is transcendent, equality is impossible.

Remark. The constant $\frac{1}{e} \approx 0.3679$ is not sharp. It is known that the best constant is between 0.4213 and 0.4232. (See I. E. Pritsker, The Gelfond–Schnirelman method in prime number theory, Canad. J. Math. 57 (2005), 1080–1101.)