IMC 2016, Blagoevgrad, Bulgaria

Day 1, July 27, 2016

Problem 1. Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Suppose that f has infinitely many zeros, but there is no $x \in (a,b)$ with $f(x) = f'(x) = 0$.

(a) Prove that $f(a)f(b) = 0$.

(b) Give an example of such a function on $[0,1]$.

(10 points)

Problem 2. Let k and n be positive integers. A sequence $(A_1, \ldots, A_k)$ of $n \times n$ real matrices is *preferred* by Ivan the Confessor if $A_i^2 \neq 0$ for $1 \leq i \leq k$, but $A_iA_j = 0$ for $1 \leq i,j \leq k$ with $i \neq j$. Show that $k \leq n$ in all preferred sequences, and give an example of a preferred sequence with $k = n$ for each n.

(10 points)

Problem 3. Let n be a positive integer. Also let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers such that $a_i + b_i > 0$ for $i = 1, 2, \ldots, n$. Prove that

$$\sum_{i=1}^{n} \frac{a_ib_i - b_i^2}{a_i + b_i} \leq \frac{\sum\limits_{i=1}^{n} a_i \cdot \sum\limits_{i=1}^{n} b_i - \left(\sum\limits_{i=1}^{n} b_i\right)^2}{\sum\limits_{i=1}^{n}(a_i + b_i)}.$$

(10 points)

Problem 4. Let $n \geq k$ be positive integers, and let $\mathcal{F}$ be a family of finite sets with the following properties:

(i) $\mathcal{F}$ contains at least $\binom{n}{k} + 1$ distinct sets containing exactly k elements;

(ii) for any two sets $A, B \in \mathcal{F}$, their union $A \cup B$ also belongs to $\mathcal{F}$.

Prove that $\mathcal{F}$ contains at least three sets with at least n elements.

(10 points)

Problem 5. Let S_n denote the set of permutations of the sequence $(1, 2, \ldots, n)$. For every permutation $\pi = (\pi_1, \ldots, \pi_n) \in S_n$, let $\mathrm{inv}(\pi)$ be the number of pairs $1 \leq i < j \leq n$ with $\pi_i > \pi_j$; i.e. the number of inversions in π. Denote by $f(n)$ the number of permutations $\pi \in S_n$ for which $\mathrm{inv}(\pi)$ is divisible by $n + 1$.

Prove that there exist infinitely many primes p such that $f(p-1) > \frac{(p-1)!}{p}$, and infinitely many primes p such that $f(p-1) < \frac{(p-1)!}{p}$.

(10 points)