IMC 2020 Online

Day 1, July 26, 2020

Problem 1. Let n be a positive integer. Compute the number of words w (finite sequences of letters) that satisfy all the following three properties:

(1) w consists of n letters, all of them are from the alphabet $\{\texttt{a}, \texttt{b}, \texttt{c}, \texttt{d}\}$;

(2) w contains an even number of letters $\texttt{a}$;

(3) w contains an even number of letters $\texttt{b}$.

(For example, for $n = 2$ there are 6 such words: `aa`, `bb`, `cc`, `dd`, `cd` and `dc`.)

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Problem 2. Let A and B be $n \times n$ real matrices such that

$$\mathrm{rk}(AB - BA + I) = 1$$

where I is the $n \times n$ identity matrix.

Prove that

$$\mathrm{trace(ABAB)} - \mathrm{trace(A^2B^2)} = \frac{1}{2}\mathrm{n(n-1)}.$$

($\mathrm{rk}(M)$ denotes the rank of matrix M, i.e., the maximum number of linearly independent columns in M. $\mathrm{trace(M)}$ denotes the trace of M, that is the sum of diagonal elements in M.)

Rustam Turdibaev, V. I. Romanovskiy Institute of Mathematics

Problem 3. Let $d \geq 2$ be an integer. Prove that there exists a constant $C(d)$ such that the following holds: For any convex polytope $K \subset \mathbb{R}^d$, which is symmetric about the origin, and any $\varepsilon \in (0,1)$, there exists a convex polytope $L \subset \mathbb{R}^d$ with at most $C(d)\varepsilon^{1-d}$ vertices such that

$$(1-\varepsilon)K \subseteq L \subseteq K.$$

(For a real α, a set $T \subset \mathbb{R}^d$ with nonempty interior is a *convex polytope with at most α vertices*, if T is a convex hull of a set $X \subset \mathbb{R}^d$ of at most α points, i.e., $T = \{\sum_{x\in X} t_x x \mid t_x \geq 0, \sum_{x\in X} t_x = 1\}$. For a real λ, put $\lambda K = \{\lambda x \mid x \in K\}$. A set $T \subset \mathbb{R}^d$ is *symmetric about the origin* if $(-1)T = T$.)

Fedor Petrov, St. Petersburg State University

Problem 4. A polynomial p with real coefficients satisfies the equation $p(x+1) - p(x) = x^{100}$ for all $x \in \mathbb{R}$. Prove that $p(1-t) \geqslant p(t)$ for $0 \leqslant t \leqslant 1/2$.

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