IMC 2020 Online

Day 1, July 26, 2020

Problem 1. Let n be a positive integer. Compute the number of words w (finite sequences of letters) that satisfy all the following three properties:

(1) w consists of n letters, all of them are from the alphabet $\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$;

(2) w contains an even number of letters $\mathtt{a}$;

(3) w contains an even number of letters $\mathtt{b}$.

(For example, for $n = 2$ there are 6 such words: aa, bb, cc, dd, cd and dc.)

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Solution 1. Let $N = \{1, 2, \ldots, n\}$. Consider a word w that satisfies the conditions and let $A, B, C, D \subset N$ be the sets of positions of letters a, b, c and d in w, respectively. By the definition of the words we have $A \sqcup B \sqcup C \sqcup D = N$. The sets A and B are constrained to have even sizes.

In order to construct all suitable words w, choose the set $S = A \cup B$ first; by the conditions, $|S| = |A| + |B|$ must be even. It is well-known that an n-element set (with $n \geq 1$) has 2^{n-1} even subsets, so there are 2^{n-1} possibilities for S.

If $S = \emptyset$ then we can choose $C \subset N$ arbitrarily, and then the set $D = S \setminus C$ is determined D uniquely. Since N has 2^n subsets, we have 2^n options for set C and therefore 2^n suitable words w with $S = \emptyset$.

Otherwise, if $k = |S| > 0$, we have to choose an arbitrary subset C of $N \setminus S$ and an even subset A of S; then $D = (N \setminus S) \setminus C$ and $B = S \setminus A$ are determined and $|B| = |S| - |A|$ will automatically be even. We have 2^{n-k} choices for C and 2^{k-1} independent choices for A; so for each nonempty even S we have $2^{n-k} \cdot 2^{k-1} = 2^{n-1}$ suitable words.

The number of nonempty even sets S is $2^{n-1} - 1$, so in total, the number of words satisfying the conditions is

$$1 \cdot 2^n + (2^{n-1} - 1) \cdot 2^{n-1} = 4^{n-1} + 2^{n-1}.$$

Solution 2. Let a_n denote the number of words of length n over $\mathcal{A} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$ such that a and b appear even number of times. Further, we define the following sequences for the number of words of length n, all over $\mathcal{A}$.

- b_n - the number of words with an odd number of a's and even number of b's
- c_n - the number of words with even number of a's and an odd number of b's
- d_n - the number of words with an odd number of a's and an odd number of b's

We will call them A-words, B-words, C-words and D-words, respectively.
It is clear that $a_1 = 2$ and that

$$a_n + b_n + c_n + d_n = 4^n.$$

First, we find a recurrence relation for a_n. If an A-word of length n begins with c or d, it can be followed by any A-word of length $n - 1$, contributing with $2a_{n-1}$. If an A-word of length n begins with a, it can be followed by any word of length $n - 1$ that contains an odd number of a's and even number of b's, thus contributing with b_{n-1}. If an A-word of length n begins with b, it can be followed by any word of length $n-1$ that contains even number of a's and an odd number of b's, thus contributing with c_{n-1}. Therefore we have the following recurrence relation:

$$a_n = 2a_{n-1} + b_{n-1} + c_{n-1}. \tag{1}$$

Next, we find a recurrence relation for b_n.
If a B-word of length n begins with c or d, it can be followed by any B-word of length $n-1$, contributing with $2b_{n-1}$. If a B-word of length n begins with a, it can be followed by any word of length $n-1$ that contains even number of a's and even number of b's, contributing with a_{n-1}. If a B-word of length n begins with b, it can be followed by any word of length $n-1$ that contains an odd number of a's and an odd number of b's, contributing with $d_{n-1} = 4^{n-1} - a_{n-1} - b_{n-1} - c_{n-1}$. Therefore we have the following recurrence relation:

$$b_n = b_{n-1} + 4^{n-1} - c_{n-1}. \tag{2}$$

Now observe that $b_k = c_k$ for all k, since simultaneously replacing a's to b's and vice versa we get a C-word from a B-word. Therefore (2) yields $b_n = 4^{n-1}$. Now (1) yields

$$a_n = 2 \cdot a_{n-1} + 2 \cdot 4^{n-2}.$$

Solving the last recurrence relation (for example, diving by 2^n we get $x_n := a_n 2^{-n}$ satisfies $x_n - x_{n-1} = 2^{n-3}$, and it remains to sum up consecutive powers of 2) we get

$$a_n = 2^{n-1} + 4^{n-1}.$$

Solution 3. Consider the sum

$$\frac{(a+b+c+d)^n + (-a-b+c+d)^n + (-a+b+c+d)^n + (a-b+c+d)^n}{4}. \tag{$*$}$$

Expanding the parentheses as

$$(a+b+c+d)^n = (a+b+c+d)(a+b+c+d)\dots(a+b+c+d),$$

we get a sum of products $x_1 \dots x_n$, $x_i \in \{a, b, c, d\}$, naturally corresponding to the words of length n over the alphabet $\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$. Consider the other terms in the numerator similarly.

If a word $x_1 \dots x_n$ contains A, B, C, D letters a,b,c and d respectively, we get $a^A b^B c^C d^D$ with the coefficient

$$\frac{1 + (-1)^{A+B} + (-1)^A + (-1)^B}{4} = \frac{(1 + (-1)^A)(1 + (-1)^B)}{4} = \begin{cases} 1, & \text{if } A \text{ and } B \text{ are even} \\ 0, & \text{otherwise.} \end{cases}$$

Hence, by substituting $a = b = c = d = 1$ in $(*)$ we get the answer $(4^n + 2^{n+1})/4 = 4^{n-1} + 2^{n-1}$.

Problem 2. Let A and B be $n \times n$ real matrices such that

$$\mathrm{rk}(AB - BA + I) = 1$$

where I is the $n \times n$ identity matrix.

Prove that

$$\mathrm{trace(ABAB) - trace(A^2B^2) = \frac{1}{2}n(n-1).}$$

($\mathrm{rk}(M)$ denotes the rank of matrix M, i.e., the maximum number of linearly independent columns in M. trace(M) denotes the trace of M, that is the sum of diagonal elements in M.)

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Solution. Let $X = AB - BA$. The first important observation is that

$$\mathrm{trace(X^2) = trace(ABAB - ABBA - BAAB + BABA) = 2trace(ABAB) - 2trace(A^2B^2)}$$

using that the trace is cyclic. So we need to prove that $\mathrm{trace(X^2) = n(n-1)}$.

By assumption, $X + I$ has rank one, so we can write $X + I = v^t w$ for two vectors v, w. So

$$X^2 = (v^t w - I)^2 = I - 2v^t w + v^t w v^t w = I + (wv^t - 2)v^t w.$$

Now by definition of X we have $\mathrm{trace(X)} = 0$ and hence $wv^t = \mathrm{trace(wv^t) = trace(v^t w) = n}$ so that indeed

$$\mathrm{trace(X^2) = n + (n-2)n = n(n-1).}$$

An alternative way to use the rank one condition is via eigenvalues: Since $X + I$ has rank one, it has eigenvalue 0 with multiplicity $n-1$. So X has eigenvalue -1 with multiplicity $n-1$. Since $\mathrm{trace(X)} = 0$ the remaining eigenvalue of X must be $n - 1$. Hence

$$\mathrm{trace(X^2) = (n-1)^2 + (n-1)\cdot 1^2 = n(n-1).}$$

Problem 3. Let $d \geq 2$ be an integer. Prove that there exists a constant $C(d)$ such that the following holds: For any convex polytope $K \subset \mathbb{R}^d$, which is symmetric about the origin, and any $\varepsilon \in (0,1)$, there exists a convex polytope $L \subset \mathbb{R}^d$ with at most $C(d)\varepsilon^{1-d}$ vertices such that

$$(1-\varepsilon)K \subseteq L \subseteq K.$$

(For a real α, a set $T \subset \mathbb{R}^d$ with nonempty interior is a *convex polytope with at most α vertices*, if T is a convex hull of a set $X \subset \mathbb{R}^d$ of at most α points, i.e., $T = \{\sum_{x\in X} t_x x \mid t_x \geq 0, \sum_{x\in X} t_x = 1\}$. For a real λ, put $\lambda K = \{\lambda x \mid x \in K\}$. A set $T \subset \mathbb{R}^d$ is *symmetric about the origin* if $(-1)T = T$.)

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Solution [in elementary terms] Let $\{p_1, \ldots, p_m\}$ be an inclusion-maximal collection of points on the boundary ∂K of K such that the homothetic copies $K_i := p_i + \frac{\varepsilon}{2}K$ have disjoint interiors. We claim that the convex hull $L := \operatorname{conv}\{p_1, \ldots, p_m\}$ satisfies all the conditions.

First, note that by convexity of K we have $aK + bK = (a+b)K$ for $a, b > 0$. It follows that $K_i \subset (1+\frac{\varepsilon}{2})K$. On the other hand, if $k \in K$, $a > 0$ and and $ak \in K_i$, then

$$p_i \in ak - \frac{\varepsilon}{2}K = ak + \frac{\varepsilon}{2}K \subset (a + \frac{\varepsilon}{2})K,$$

and since p_i is a boundary point of K, we get $a + \frac{\varepsilon}{2} \geqslant 1$, $a \geqslant 1 - \frac{\varepsilon}{2}$. It means that all K_i lie between $(1-\frac{\varepsilon}{2})K$ and $(1+\frac{\varepsilon}{2})K$. Since their interiors are disjoint, by the volume counting we obtain

$$m\left(\frac{\varepsilon}{2}\right)^d \leqslant \left(1+\frac{\varepsilon}{2}\right)^d - \left(1-\frac{\varepsilon}{2}\right)^d \leqslant (3/2)^d\varepsilon$$

(since $F(\varepsilon) = (1+\frac{\varepsilon}{2})^d - (1-\frac{\varepsilon}{2})^d$ is a polynomial in ε without constant term with non-negative coefficients which sum up to $(3/2)^d - (1/2)^d$), therefore $m \leqslant 3^d\varepsilon^{1-d}$.

It is clear that $L \subseteq K$, so it remains to prove that $(1-\varepsilon)K \subseteq L$. Assume the contrary: there exists a point $p \in (1-\varepsilon)K \setminus L$. Separate p from L by a hyperplane: Choose a linear functional ℓ such that $\ell(p) > \max_{x\in L} \ell(x) = \max_i \ell(p_i)$. Choose $x \in K$ such that $\ell(x) =: a$ is maximal possible. Note that by our construction $x + \frac{\varepsilon}{2}K$ has a common point with some K_i: there exists a point $z \in (x + \frac{\varepsilon}{2}K) \cap (p_i + \frac{\varepsilon}{2}K)$. We have

$$\ell(p_i) + \frac{\varepsilon}{2}a \geqslant \ell(z) \geqslant \ell(x) - \frac{\varepsilon}{2}a,$$

and therefore $\ell(p_i) \geqslant a(1-\varepsilon)$. Since $p \in (1-\varepsilon)K$, we obtain $\ell(p) \leqslant a(1-\varepsilon)$. A contradiction.

Solution [in the language of Banach spaces] Equip $\mathbb{R}^d$ with the norm $\|\cdot\|$, whose unit ball is K, call this Banach space V. Choose an inclusion maximal set $X \subset \partial K$ whose pairwise distances are $\geq \varepsilon$. Put $L = \operatorname{conv} X$.

The inclusion $L \subseteq K$ follows from the convexity of K. If the inclusion $(1-\varepsilon)K \subseteq L$ fails then the Hahn–Banach theorem provides a unit linear functional $\lambda \in V^*$ such that $\max\{\lambda(L)\} = \max\{\lambda X\} \leq 1-\varepsilon$. Then the point $x \in K$, where the maximum $\max\{\lambda(K)\} = 1$ is attained (thanks to the finite dimension and compactness) is in ∂K and, as λ witnesses, at distance $\geq \varepsilon$ from all other points of L and X, contradicting the inclusion-maximality of X.

The upper bound for the cardinality $|X|$ is obtained by noting that the $\varepsilon/2$ balls centered at the points of X are pairwise disjoint and lie in the difference of balls $(1+\varepsilon/2)K \setminus (1-\varepsilon/2)K$, whose volume is $\left((1+\varepsilon/2)^d - (1-\varepsilon/2)^d\right)\operatorname{vol}K$, the volume of each of the small balls being $\varepsilon^d/2^d \operatorname{vol}K$. Hence

$$|X| \leq \frac{(2+\varepsilon)^d - (2-\varepsilon)^d}{\varepsilon^d} = O(\varepsilon^{1-d}).$$

Problem 4. A polynomial p with real coefficients satisfies the equation $p(x+1)-p(x)=x^{100}$ for all $x\in\mathbb{R}$. Prove that $p(1-t)\geqslant p(t)$ for $0\leqslant t\leqslant 1/2$.

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Solution 1. Denote $h(z)=p(1-\bar{z})-p(z)$ for complex z. For $t\in\mathbb{R}$ we have $h(it)=p(1+it)-p(it)=t^{100}$, $h(1/2+it)=0$.

If $p(z)=c_nz^n+\ldots+c_0, c_n\neq 0$, we have

$$h(a+it)=p((1-a)+it)-p(a+it)=(1-2a)\left(nc_ni^{n-1}t^{n-1}+Q(t,a)\right)$$

for some polynomial Q having degree at most $n-2$ with respect to the variable t. Substituting $a=0$ we get $n=101$, $c_n=1/101$.

Next, for large $|t|$ we see that $\Re(h(a+it))>0$ for $0\leqslant a<1/2$.

Therefore by Maximum Principle for the harmonic function $\Re h$ and the rectangle $[0,1/2]\times[-N,N]$ for large enough N we conclude that $\Re h$ is non-negative in this rectangle, in particular on $[0,1/2]$, as we need.

Solution 2. Let $p(x)=\sum_{j=0}^m a_jx^j$. Then

$$p(x+1)-p(x)=\sum_{j=0}^m a_j\left((x+1)^j-x^j\right)=a_1+a_2(2x+1)+\cdots+a_m\left(mx^{m-1}+\binom{m}{2}x^{m-2}+\cdots+1\right).$$

This implies that $m=101$, $ma_m=1$ so $a_{101}=\frac{1}{101}$, $(m-1)a_{m-1}+a_m\binom{m}{2}=0$ so $a_{100}=-\frac{1}{2}$ etc. For $j\geqslant 1$ a_j is uniquely defined, a_0 may be chosen arbitrarily.

The equality $p_{2n}(\frac{1}{2})=0$ holds because $0=p_{2n}(\frac{1}{2})+p_{2n}(1-\frac{1}{2})=2p_{2n}(\frac{1}{2})$. Let $n\geqslant 1$ be an integer and let p_n be a polynomial such that $p_n(x+1)-p_n(x)=x^n$ for all x and $p_n(0)=0=p_n(1)$. The above considerations prove the uniqueness of p_n. We have $p_1(x)=\frac{1}{2}x^2-\frac{1}{2}x$. Also $p_n'(x+1)-p_n'(x)=nx^{n-1}=n\left(p_{n-1}(x+1)-p_{n-1}(x)\right)$. Therefore $p_n'(x)=np_{n-1}(x)+c_{n-1}$ for a properly chosen constant c_{n-1}. We shall prove that

$$(1)\quad p_{2n-1}(x)-p_{2n-1}(1-x)=0,\ p_{2n}(x)+p_{2n}(1-x)=0,\ c_{2n}=0,\ p_{2n}''(x)=2n(2n-1)p_{2n-2}(x)$$

for $n=1,2,\ldots$ and for all x. Simple computation shows that $p_1(x)-p_1(1-x)=0$. We have $(p_2(x)+p_2(1-x))'=2p_1(x)+c_1-(2p_1(1-x)+c_1)=0$ so the map $x\mapsto p_2(x)+p_2(1-x)$ is constant thus $p_2(x)+p_2(1-x)=p_2(0)+p_2(1-0)=0$. If the first two equalities hold for some n then $(p_{2n+1}(x)-p_{2n+1}(1-x))'=(2n+1)p_{2n}(x)+c_{2n}+(p_{2n}(1-x)+c_{2n})=2c_{2n}$ so there exists $b\in R$ such that $p_{2n+1}(x)-p_{2n+1}(1-x)=2c_{2n}x+b$ for all x. $p_{2n+1}(0)-p_{2n+1}(1-0)=0$ and $p_{2n+1}(1)-p_{2n+1}(1-1)=0$ so $2c_{2n}=0=b$. This proves that $p_{2n+1}(x)-p_{2n+1}(1-x)=0$ for all x. In a similar way we shall prove the second equality: $(p_{2n+2}(x)+p_{2n+2}(1-x))'=(2n+2)p_{2n+1}'(x)+c_{2n+1}-(2n+2)\left(p_{2n+1}(1-x)+c_{2n+1}\right)=0$ so the map $x\mapsto p_{2n+2}(x)+p_{2n+2}(1-x)$ is constant hence $p_{2n+2}(x)+p_{2n+2}(1-x)=p_{2n+2}(0)+p_{2n+2}(1-0)=0$ for all x. Now $p_{2n+2}''(x)=((2n+2)p_{2n+1}(x)+c_{2n+1})'=(2n+2)p_{2n+1}'(x)=(2n+2)((2n+1)p_{2n}(x)+c_{2n})=(2n+2)(2n+1)p_{2n}(x)$. Since $p_2'(x)=2p_1(x)+c_1=x^2-x+c_1$ we obtain $p_2''(x)=2x-1<0$ for $x<\frac{1}{2}$.The function p_2 is strictly concave on $[0,\frac{1}{2}]$ and $p_2(0)=0=p_2(\frac{1}{2})$. Therefore $p_2(x)>0$ for $x\in(0,\frac{1}{2})$. This together with the equality $p_4(x)=12p_2(x)$ implies that p_4 is strictly convex on $[0,\frac{1}{2}]$ so in view of $p_4(0)=0=p_4(\frac{1}{2})$ we conclude that $p_4(x)<0$ for $x\in(0,\frac{1}{2})$. Easy induction shows that for $x\in(0,\frac{1}{2})$ one has $p_{2n}(x)>0$ for an odd n and $p_{2n}(x)<0$ for an even n. If $t\in(0,\frac{1}{2})$ then by (1) we get $p_{100}(1-t)-p_{100}(t)=-2p_{100}(t)>0$ as required.