IMC 2021 Online

First Day, August 3, 2021

Solutions

Problem 1. Let A be a real $n \times n$ matrix such that $A^3 = 0$.

(a) Prove that there is a unique real $n \times n$ matrix X that satisfies the equation

$$X + AX + XA^2 = A.$$

(b) Express X in terms of A.

(proposed by Bekhzod Kurbonboev, Institute of Mathematics, Tashkent)

Hint: (a) Multiply the equation by some power of A from left and another power of A from right.

(b) Substitute repeatedly $X = A - AX - XA^2$.

Solution 1. First suppose that some matrix X satisfies the equation. We can obtain new equations if we multiply the given equation by some power of A from left and another power of A from right. For example,

$$A^2(X + AX + XA^2)A^2 = A^2XA^2 + A^3 \cdot XA^2 + A^2XA \cdot A^3 = A^2XA^2.$$

The right-hand side is $A^2 \cdot A \cdot A^2 = A^3 \cdot A^2 = 0$, so

$$\begin{aligned} A^2XA^2 &= A^2(X + AX + XA^2)A^2 = A^5 = 0. \qquad \text{Similarly,} \\ A^2X &= A^2(X + AX + XA^2) = A^3 = 0 \\ AXA &= A(X + AX + XA^2)A = A^3 = 0 \\ XA^2 &= (X + AX + XA^2)A^2 = A^3 = 0 \\ AX &= A(X + AX + XA^2)A = A^2. \qquad \text{Finally} \\ X &= A - AX - XA^2 = A - A^2. \end{aligned}$$

Hence, no matrix other than $A - A^2$ can satisfy the equation.

Note that the argument above does not prove that the matrix $X = A - A^2$ satisfies the equation, because the steps cannot be done in reverse order. That must be verified separately. Indeed,

$$X + AX + XA^2 = (A - A^2) + A(A - A^2) + (A - A^2)A^2 = A - A^4 = A.$$

Hence, $X = A - A^2$ is the unique solution of the equation.

Remark. By multiplying the equation by A^n from left and by A^k from right we can get 9 different equations:

$$\begin{array}{ccc} X + AX + XA^2 = A & XA + AXA = A^2 & XA^2 + AXA^2 = 0 \\ AX + A^2X + AXA^2 = A^2 & AXA + A^2XA = 0 & AXA^2 + A^2XA^2 = 0 \\ A^2X + A^2XA^2 = 0 & A^2XA = 0 & A^2XA^2 = 0 \end{array}$$

These formulas provide a system of linear equations for the nine matrices X, AX, A^2X, XA, AXA, A^2XA, XA^2, AXA^2 and A^2XA^2.

Solution 2. We use a different approach to express X in terms of A. If some matrix X satisfies the equation then

$$X = A - AX - XA^2.$$

Let us substitute this identity in the right-hand side repeatedly until X cancels out everywhere. Notice that by the condition $A^3 = 0$ we have $A^3 = A^4 = A^5 = A^3X = XA^4 = AXA^4 = A^3XA^2 = 0$, so

$$\begin{aligned}
X &= A - AX - XA^2 \\
&= A - A(A - AX - XA^2) - (A - AX - XA^2)A^2 \\
&= A - (A^2 - A^2X - AXA^2) - (A^3 - AXA^2 - XA^4) \\
&= A - A^2 + A^2X + 2AXA^2 \\
&= A - A^2 + A^2(A - AX - XA^2) + 2A(A - AX - XA^2)A^2 \\
&= A - A^2 + (A^3 - A^3X - A^2XA^2) + 2(A^4 - A^2XA^2 - AXA^4) \\
&= A - A^2 - 3A^2XA^2 \\
&= A - A^2 - 3A^2(A - AX - XA^2)A^2 \\
&= A - A^2 - 3(A^5 - A^3XA^2 - A^2XA^4) \\
&= A - A^2.
\end{aligned}$$

To complete the solution, we have to verify that $X = A - A^2$ is indeed a solution. This step is the same as in Solution 1.

Solution 3. Let $B = I - A + A^2$ so that B is the inverse of $I + A$. Multiplying by B from the left, the equation is equivalent to

$$X + BXA^2 = BA. \tag{1}$$

Now assume X satisfies the equation. Multiplying by A^2 from the right and using $A^3 = 0$ we get $XA^2 = 0$. Hence the equation simplifies to $X = BA = A - A^2$.

On the other hand, $X = BA$ obviously satisfies (1).

Problem 2. Let n and k be fixed positive integers, and let a be an arbitrary non-negative integer. Choose a random k-element subset X of $\{1, 2, \dots, k + a\}$ uniformly (i.e., all k-element subsets are chosen with the same probability) and, independently of X, choose a random n-element subset Y of $\{1, \dots, k + n + a\}$ uniformly.

Prove that the probability

$$\mathsf{P}\Big(\min(Y) > \max(X)\Big)$$

does not depend on a.

(proposed by Fedor Petrov, St. Petersburg State University)

Hint: The sets X and Y with $\min(Y) > \max(X)$ are uniquely determined by $X \cup Y$.

Solution 1. The number of choices for (X, Y) is $\binom{k+a}{k} \cdot \binom{n+k+a}{n}$.

The number of such choices with $\min(Y) > \max(X)$ is equal to $\binom{n+k+a}{n+k}$ since this is the number of choices for the $n+k$-element set $X \cup Y$ and this union together with the condition $\min(Y) > \max(X)$ determines X and Y uniquely (note in particular that no elements of X will be larger than $k + a$). Hence the probability is

$$\frac{\binom{n+k+a}{n+k}}{\binom{k+a}{k} \cdot \binom{n+k+a}{n}} = \frac{1}{\binom{n+k}{k}}$$

where the identity can be seen by expanding the binomial coefficients on both sides into factorials and canceling.

Since the right hand side is independent of a, the claim follows.

Solution 2. Let f be the increasing bijection from $\{1,2,\dots,k+a\}$ to $\{1,\dots,k+a+n\} \setminus Y$. Note that $\min(Y) > \max(X)$ if and only if $\min(Y) > \max(f(X))$.

Note that

$$\{Z_n := Y, Z_k := f(X), Z_a := f(\{1,2,\dots,k+a\} \setminus X)\}$$

is a random partition of

$$\{1,\dots,n+k+a\} = Z_n \sqcup Z_k \sqcup Z_a$$

into an n-subset, k-subset, and a-subset.

If an a-subset Z_a is fixed, the conditional probability that $\min(Z_k) > \max(Z_n)$ always equals $1/\binom{n+k}{k}$. Therefore the total probability also equals $1/\binom{n+k}{k}$.

Problem 3. We say that a positive real number d is *good* if there exists an infinite sequence $a_1, a_2, a_3, \ldots \in (0,d)$ such that for each n, the points $a_1,\dots,a_n$ partition the interval $[0,d]$ into segments of length at most $1/n$ each. Find

$$\sup\left\{d \;\middle|\; d \text{ is good}\right\}.$$

(proposed by Josef Tkadlec)

Hint: To get an upper bound, use that some of the gaps after n steps are still intact some steps later.

Solution. Let $d^\star = \sup\{d \mid d \text{ is good}\}$. We will show that $d^\star = \ln(2) \doteq 0.693$.

1. $d^\star \le \ln 2$:

   Assume that some d is good and let $a_1, a_2, \ldots$ be the witness sequence.

   Fix an integer n. By assumption, the prefix $a_1,\dots,a_n$ of the sequence splits the interval $[0,d]$ into $n+1$ parts, each of length at most $1/n$.

   Let $0 \le \ell_1 \le \ell_2 \le \cdots \le \ell_{n+1}$ be the lengths of these parts. Now for each $k = 1,\dots,n$ after placing the next k terms $a_{n+1},\dots,a_{n+k}$, at least $n+1-k$ of these initial parts remain intact. Hence $\ell_{n+1-k} \le \frac{1}{n+k}$. Hence

   $$d = \ell_1 + \cdots + \ell_{n+1} \le \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}. \tag{2}$$

   As $n \to \infty$, the RHS tends to $\ln(2)$ showing that $d \le \ln(2)$.

   Hence $d^\star \le \ln 2$ as desired.

2. $d^\star \ge \ln 2$:

   Observe that

   $$\ln 2 = \ln 2n - \ln n = \sum_{i=1}^{n} \ln(n+i) - \ln(n+i-1) = \sum_{i=1}^{n} \ln\left(1 + \frac{1}{n+i-1}\right).$$

   Interpreting the summands as lengths, we think of the sum as the lengths of a partition of the segment $[0, \ln 2]$ in n parts. Moreover, the maximal length of the parts is $\ln(1+1/n) < 1/n$.

   Changing n to $n+1$ in the sum keeps the values of the sum, removes the summand $\ln(1+1/n)$, and adds two summands

   $$\ln\left(1+\frac{1}{2n}\right) + \ln\left(1+\frac{1}{2n+1}\right) = \ln\left(1+\frac{1}{n}\right).$$

This transformation may be realized by adding one partition point in the segment of length $\ln(1+1/n)$.

In total, we obtain a scheme to add partition points one by one, all the time keeping the assumption that once we have $n-1$ partition points and n partition segments, all the partition segments are smaller than $1/n$.

The first terms of the constructed sequence will be $a_1 = \ln\frac{3}{2}, a_2 = \ln\frac{5}{4}, a_3 = \ln\frac{7}{4}, a_4 = \ln\frac{9}{8}, \dots$.

Remark. This remark describes in fact the same solution from a different view and some ideas behind it. It could be erased after marking is finished. Estimate (2) is quite natural. To prove that RHS tends to $\ln 2$ we use some integral estimates by

$$\int_n^{2n+1} \frac{1}{x} dx = \ln(2n+1) - \ln n.$$

Here we can observe that

$$\int_n^{2n} \frac{1}{x} dx = \ln 2$$

is independent of n. This can help us with the construction since the above equality means

$$I_1 = \int_n^{n+1} \frac{1}{x} dx = \int_{2n}^{2n+1} \frac{1}{x} dx + \int_{2n+1}^{2n+2} \frac{1}{x} dx = I_2 + I_3,$$

so, interval of length I_1 can be splitted into two intervals of lengths I_2 and I_3. In fact, after placing the point a_n in the construction for $d = \ln 2$, the lengths of the $n+1$ intervals are

$$\int_{n+1}^{n+2} \frac{1}{x}, \int_{n+2}^{n+3} \frac{1}{x}, \dots, \int_{2n+1}^{2n+2} \frac{1}{x}$$

with total length

$$d = \int_{n+1}^{2n+2} \frac{1}{x} = \ln 2.$$

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose that for every $\varepsilon > 0$, there exists a function $g : \mathbb{R} \to (0, \infty)$ such that for every pair (x, y) of real numbers,

$$\text{if} \quad |x - y| < \min\{g(x), g(y)\}, \quad \text{then} \quad |f(x) - f(y)| < \varepsilon.$$

Prove that f is the pointwise limit of a sequence of continuous $\mathbb{R} \to \mathbb{R}$ functions, i.e., there is a sequence $h_1, h_2, \dots$ of continuous $\mathbb{R} \to \mathbb{R}$ functions such that $\lim_{n\to\infty} h_n(x) = f(x)$ for every $x \in \mathbb{R}$.

(proposed by Camille Mau, Nanyang Technological University, Singapore)

Hint: Start from a segment in place of $\mathbb{R}$ and use its compactness. Or recall the cool things called "the Lebesgue characterization theorem" and "the Baire characterization theorem".

Solution 1. Since g depends also on ε, let us use the notation $g(x, \varepsilon)$. Considering only $\varepsilon = 1/n$ for positive integer n will suffice to reach our conclusions, hence we may use $\min\{g(x, 1/m) \mid m \le n\}$ in place of $g(x, 1/n)$ and thus assume $g(x, \varepsilon)$ decreasing in ε.

For any $x \in \mathbb{R}$, choose $\delta_n(x) = \min\{1/n, g(x, 1/n)\}$. Of the $\delta_n(x)$-neighborhoods of all x select (using local compactness of the reals) an inclusion-minimal locally finite covering $\{U_i\}$. From its inclusion-minimality it follows that we may enumerate U_i with $i \in \mathbb{Z}$ so that $U_i \cap U_j \ne \varnothing$ only when $|i - j| \le 1$ and the enumeration goes from left to right on the real line. For an assumed n, let x_i be the center of U_i and $\delta_i = \delta_n(x_i)$, so that $U_i = (x_i - \delta_i, x_i + \delta_i)$ and $\delta_i < 1/n$ for all i.

Now define a continuous $f_n : \mathbb{R} \to \mathbb{R}$ so that it equals $f(x_i)$ in $U_i \setminus (U_{i-1} \cup U_{i+1})$, and so that f_n changes continuously between $f(x_{i-1})$ and $f(x_i)$ in the intersection $U_{i-1} \cap U_i$.

Now we show that $f_n \to f$ pointwise. Fix a point x and $\varepsilon = 1/m > 0$, and choose

$$n > \max\{1/g(x, \varepsilon), m\}.$$

Examine the construction of f_n for any such n. Observe that $g(x, \varepsilon) > 1/n > \delta_i$ and $1/n < 1/m$. There are two cases:

- x belongs to the unique U_i. Then using the monotonicity of $g(x,\varepsilon)$ in ε we have

$$|x_i - x| < \delta_i \le \min\left\{g\left(x_i, \frac{1}{n}\right), g\left(x,\varepsilon\right)\right\} \le \min\left\{g(x_i,\varepsilon), g(x,\varepsilon)\right\}.$$

Hence

$$|f(x) - f_n(x)| = |f(x) - f(x_i)| < \varepsilon.$$

- x belongs to $U_{i-1} \cap U_i$. Similar to the previous case,

$$|f(x) - f(x_{i-1})|, |f(x) - f(x_i)| < \varepsilon.$$

Since $f_n(x)$ is between $f_n(x_{i-1}) = f(x_{i-1})$ and $f_n(x_i) = f(x_i)$ by construction, we have

$$|f(x) - f_n(x)| < \varepsilon.$$

We have that $|f(x) - f_n(x)| < \varepsilon$ holds for sufficiently large n, which proves the pointwise convergence.

Solution 2. This solution uses the Baire characterization theorem: *A function $f : \mathbb{R} \to \mathbb{R}$ is a pointwise limit of continuous functions if and only if its restriction to every non-empty closed subset of $\mathbb{R}$ has a point of continuity.*

Assume the contrary in view of the above theorem: $A \subseteq \mathbb{R}$ is a non-empty closed set and f has no point of continuity in A. Let's think that f is defined only on A.

Then for all $x \in A$ there exist rationals $p < q$ for which $\limsup_x f > q$, $\liminf_x f < p$. Apply the Baire category theorem: *If a complete metric space A is a countable union of sets then some of the sets is dense in a positive radius metric ball of A.* It follows that there exist p and q, which serve for a subset $B \subset A$ which is dense on a certain ball (in the induced metric of the real line) $A_1 \subset A$. It yields that both sets $Q = f^{-1}(q,\infty)$ and $P = f^{-1}(-\infty,p)$ are dense in A_1.

Choose $\varepsilon = (q-p)/10$ and find k for which the set $S = \{x : g(x) > 1/k\}$ is also dense on a certain ball $A_2 \subset A_1$. Partition S into subsets where $f(x) > (p+q)/2$ and $f(x) \leqslant (p+q)/2$, one of them is again dense somewhere in A_3, say the latter.

Now take any point $y \in A_3 \cap Q$ and a very close (within distance $\min(1/k, g(y))$) to y point x with $g(x) > 1/k$ but $f(x) \leqslant (p+q)/2$. This pair x, y contradicts the property of f from the problem statement.

Solution 3. This solution uses the Lebesgue characterization theorem: *If $f : \mathbb{R} \to \mathbb{R}$ is a function and, for all real c, the sublevel and superlevel sets $\{x \mid f(x) \geqslant c\}$, $\{x \mid f(x) \leqslant c\}$ are countable intersections of open sets then f is a pointwise limit of continuous functions.*

Now the solution follows from the formula with a countable intersection of the unions of intervals:

$$\{x \mid f(x) \geqslant c\} = \bigcap_{n,k=1}^{\infty} \bigcup_{\substack{y\in\mathbb{R} \\ f(y)\geqslant c}} \left(y - \min\left\{\frac{1}{k}, g\left(y, \frac{1}{n}\right)\right\}, y + \min\left\{\frac{1}{k}, g\left(y, \frac{1}{n}\right)\right\}\right) \qquad (*)$$

and the similar formula for $\{x : f(x) \leqslant c\}$. It remains to prove $(*)$.

The left hand side is obviously contained in the right hand side, just put $y = x$.

To prove the opposite inclusion assume the contrary, that $f(x) < c$, but x is contained in the right hand side. Choose a positive integer n such that $f(x) < c - 1/n$ and k such that $g(x, 1/n) > 1/k$. Then, since x belongs to the right hand side, we see that there exists y such that $f(y) \geqslant c$ and

$$|x - y| < \min\left\{g\left(y, \frac{1}{n}\right), \frac{1}{k}\right\} \leqslant \min\left\{g\left(y, \frac{1}{n}\right), g\left(x, \frac{1}{n}\right)\right\},$$

which yields $f(x) \geqslant f(y) - 1/n \geqslant c - 1/n$, a contradiction.