IMC 2021 Online

Second Day, August 4, 2021

Problem 5. Let A be a real $n \times n$ matrix and suppose that for every positive integer m there exists a real symmetric matrix B such that

$$2021B = A^m + B^2.$$

Prove that $|\det A| \leq 1$.

(10 points)

Problem 6. For a prime number p, let $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ be the group of invertible 2×2 matrices of residues modulo p, and let S_p be the symmetric group (the group of all permutations) on p elements. Show that there is no injective group homomorphism $\varphi : \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z}) \to S_p$.

(10 points)

Problem 7. Let $D \subseteq \mathbb{C}$ be an open set containing the closed unit disk $\{z : |z| \leq 1\}$. Let $f : D \to \mathbb{C}$ be a holomorphic function, and let $p(z)$ be a monic polynomial. Prove that

$$|f(0)| \leq \max_{|z|=1} |f(z)p(z)|.$$

(10 points)

Problem 8. Let n be a positive integer. At most how many distinct unit vectors can be selected in $\mathbb{R}^n$ such that from any three of them, at least two are orthogonal?

(10 points)