IMC 2021 Online

Second Day, August 4, 2021

Solutions

Problem 5. Let A be a real $n \times n$ matrix and suppose that for every positive integer m there exists a real symmetric matrix B such that

$$2021B = A^m + B^2.$$

Prove that $|\det A| \le 1$.

(proposed by Rafael Filipe dos Santos, Instituto Militar de Engenharia, Rio de Janeiro)

Hint: The determinant is the product of the eigenvalues.

Solution. Let B_m be the corresponding matrix B depending on m:

$$2021B_m = A^m + B_m^2.$$

For $m = 1$, we obtain $A = 2021B_1 - B_1^2$. Since B_1 is real and symmetric, so is A. Thus A is diagonalizable and all eigenvalues of A are real.

Now fix a positive integer m and let λ be any real eigenvalue of A. Considering the diagonal form of both A and B_m, we know that there exists a real eigenvalue μ of B_m such that

$$2021\mu = \lambda^m + \mu^2 \Rightarrow \mu^2 - 2021\mu + \lambda^m = 0.$$

The last equation is a second degree equation with a real root. Therefore, the discriminant is non-negative:

$$2021^2 - 4\lambda^m \ge 0 \Rightarrow \lambda^m \le \frac{2021^2}{4}.$$

If $|\lambda| > 1$, letting m even sufficiently large we reach a contradiction. Thus $|\lambda| \le 1$.

Finally, since $\det A$ is the product of the eigenvalues of A and each of them has absolute value less then or equal to 1, we get $|\det A| \le 1$ as desired.

Solution. Different solution can be found in paper s2002

Problem 6. For a prime number p, let $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ be the group of invertible 2×2 matrices of residues modulo p, and let S_p be the symmetric group (the group of all permutations) on p elements. Show that there is no injective group homomorphism $\varphi : \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z}) \to S_p$.

(proposed by Thiago Landim, Sorbonne University, Paris)

Hint: First find what the monomorphism must do with elements of order p.

Solution. For $p = 2$, just note that $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$ has more than $2 = |S_2|$ elements.

From now on, let p be an odd prime and suppose that there exists such a homomorphism.

The matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

has order p and commutes with the matrix

$$B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

of order 2, hence AB has order $2p$. But there is no permutation in S_p of order $2p$ since only p-cycles have order divisible by p, and their order is exactly p.

Problem 7. Let $D \subseteq \mathbb{C}$ be an open set containing the closed unit disk $\{z \ : \ |z| \le 1\}$. Let $f : D \to \mathbb{C}$ be a holomorphic function, and let $p(z)$ be a monic polynomial. Prove that

$$\left|f(0)\right| \le \max_{|z|=1} \left|f(z)p(z)\right|.$$

(proposed by Lars Hörmander)

Hint: Apply the maximum principle or the Cauchy formula to a suitable function $f(z)q(z)$.

Solution.

Let $q(z) = z^n \cdot \overline{p\left(1/\overline{z}\right)}$, or more explicitly, if

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0,$$

let

$$q(z) = 1 + \overline{a_{n-1}}z + \cdots + \overline{a_0}z^n.$$

Note that for $|z| = 1$ we have $1/\overline{z} = z$ and hence $|q(z)| = |p(z)|$. Hence by the maximum principle or the Cauchy formula for the product of f and q, it follows that

$$|f(0)| = |f(0)q(0)| \le \max_{|z|=1} |f(z)q(z)| = \max_{|z|=1} |f(z)p(z)|.$$

Problem 8. Let n be a positive integer. At most how many distinct unit vectors can be selected in $\mathbb{R}^n$ such that from any three of them, at least two are orthogonal?

(proposed by Alexander Polyanskii, Moscow Institute of Physics and Technology; based on results of Paul Erdős and Moshe Rosenfeld)

Hint: Play with the Gram matrix of these vectors.

Solution 1. $2n$ is the maximal number.

An example of $2n$ vectors in the set is given by a basis and its opposite vectors. In the rest of the text we prove that it is impossible to have $2n+1$ vectors in the set.

Consider the Gram matrix A with entries $a_{ij} = e_i \cdot e_j$. Its rank is at most n, its eigenvalues are real and non-negative. Put $B = A - I_{2n+1}$, this is the same matrix, but with zeros on the diagonal. The eigenvalues of B are real, greater or equal to -1, and the multiplicity of -1 is at least $n+1$.

The matrix $C = B^3$ has the following diagonal entries

$$c_{ii} = \sum_{i \ne j \ne k \ne i} a_{ij}a_{jk}a_{ki}.$$

The problem statement implies that in every summand of this expression at least one factor is zero. Hence $\operatorname{tr} C = 0$. Let $x_1, \ldots, x_m$ be the positive eigenvalues of B, their number is $m \le n$ as noted above. From $\operatorname{tr} B = \operatorname{tr} C$ we deduce (taking into account that the eigenvalues between -1 and 0 satisfy $\lambda^3 \ge \lambda$):

$$x_1 + \cdots + x_m \ge x_1^3 + \cdots + x_m^3$$

Applying $\operatorname{tr} C = 0$ once again and noting that C has eigenvalue -1 of multiplicity at least $n+1$, we obtain

$$x_1^3 + \cdots + x_m^3 \geq n + 1.$$

It also follows that

$$(x_1 + \cdots + x_m)^3 \geq \left(x_1^3 + \cdots + x_m^3\right)(n+1)^2.$$

By Hölder's inequality, we obtain

$$\left(x_1^3 + \cdots + x_m^3\right) m^2 \geq (x_1 + \cdots + x_m)^3,$$

which is a contradiction with $m \leq n$.

Solution 2. Let P_i denote the projection onto i-th vector, $i = 1, \ldots, N$. Then our relation reads as $\operatorname{tr}(P_i P_j P_k) = 0$ for distinct i, j, k. Consider the operator $Q = \sum_{i=1}^N P_i$, it is non-negative definite, let $t_1, \ldots, t_n$ be its eigenvalues, $\sum t_i = \operatorname{tr} Q = N$. We get

$$\sum t_i^3 = \operatorname{tr} Q^3 = N + 6 \sum_{i<j} \operatorname{tr} P_i P_j = N + 3(\operatorname{tr} Q^2 - N) = 3 \sum t_i^2 - 2N$$

(we used the obvious identities like $\operatorname{tr} P_i P_j P_i = \operatorname{tr} P_i^2 P_j = \operatorname{tr} P_i P_j$). But $(t_i - 2)^2(t_i + 1) = t_i^3 - 3t_i^2 + 4 \geqslant 0$, thus $-2N = \sum t_i^3 - 3t_i^2 \geqslant -4n$ and $N \leqslant 2n$.