IMC 2022

First Day, August 3, 2022

Solutions

Problem 1. Let $f : [0,1] \to (0,\infty)$ be an integrable function such that $f(x) \cdot f(1-x) = 1$ for all $x \in [0,1]$. Prove that

$$\int_0^1 f(x)\ \mathrm{d}x \geq 1.$$

(proposed by Mike Daas, Universiteit Leiden)

Hint: Apply the AM–GM inequality.

Solution 1. By the AM–GM inequlity we have

$$f(x) + f(1-x) \geq 2\sqrt{f(x)f(1-x)} = 2.$$

By integrating in the interval $[0, \frac{1}{2}]$ we get

$$\int_0^1 f(x)\mathrm{d}x = \int_0^{\frac{1}{2}} f(x)\mathrm{d}x + \int_0^{\frac{1}{2}} f(1-x)\mathrm{d}x = \int_0^{\frac{1}{2}} \big(f(x) + f(1-x)\big)\mathrm{d}x \geq \int_0^{\frac{1}{2}} 2\mathrm{d}x = 1.$$

Solution 2. From the condition, we have

$$\int_0^1 f(x)dx = \int_0^1 f(1-x)dx = \int_0^1 \frac{1}{f(x)}dx$$

and hence, using the positivity of f, the claim follows since

$$\left(\int_0^1 f(x)dx\right)^2 = \int_0^1 f(x)dx \cdot \int_0^1 \frac{1}{f(x)}dx \geq \left(\int_0^1 1dx\right)^2 \geq 1$$

by the Cauchy-Schwarz inequality.

Problem 2. Let n be a positive integer. Find all $n \times n$ real matrices A with only real eigenvalues satisfying

$$A + A^k = A^T$$

for some integer $k \geq n$.

(A^T denotes the transpose of A.)

(proposed by Camille Mau, Nanyang Technological University)

Hint: Consider the eigenvalues of A.

Solution 1. Taking the transpose of the matrix equation and substituting we have

$$A^T + (A^T)^k = A \implies A + A^k + (A + A^k)^k = A \implies A^k(I + (I + A^{k-1})^k) = 0.$$

Hence $p(x) = x^k(1 + (1 + x^{k-1})^k)$ is an annihilating polynomial for A. It follows that all eigenvalues of A must occur as roots of p (possibly with different multiplicities). Note that for all $x \in \mathbb{R}$ (this can be seen by considering even/odd cases on k),

$$(1 + x^{k-1})^k \geq 0,$$

and we conclude that the only eigenvalue of A is 0 with multiplicity n.

Thus A is nilpotent, and since A is $n \times n$, $A^n = 0$. It follows $A^k = 0$, and $A = A^T$. Hence A can only be the zero matrix: A is real symmetric and so is orthogonally diagonalizable, and all its eigenvalues are 0.

Remark. It's fairly easy to prove that eigenvalues must occur as roots of any annihilating polynomial. If λ is an eigenvalue and v an associated eigenvector, then $f(A)v = f(\lambda)v$. If f annihilates A, then $f(\lambda)v = 0$, and since $v \neq 0$, $f(\lambda) = 0$.

Solution 2. If λ is an eigenvalue of A, then $\lambda + \lambda^k$ is an eigenvalue of $A^T = A + A^k$, thus of A too. Now, if k is odd, then taking λ with maximal absolute value we get a contradiction unless all eigenvalues are 0. If k is even, the same contradiction is obtained by comparing the traces of A^T and $A + A^k$.

Hence all eigenvalues are zero and A is nilpotent. The hypothesis that $k \geq n$ ensures $A = A^T$. A nilpotent self-adjoint operator is diagonalizable and is necessarily zero.

Problem 3. Let p be a prime number. A flea is staying at point 0 of the real line. At each minute, the flea has three possibilities: to stay at its position, or to move by 1 to the left or to the right. After $p-1$ minutes, it wants to be at 0 again. Denote by $f(p)$ the number of its strategies to do this (for example, $f(3) = 3$: it may either stay at 0 for the entire time, or go to the left and then to the right, or go to the right and then to the left). Find $f(p)$ modulo p.

(proposed by Fedor Petrov, St. Petersburg)

Hint: Find a recurrence for $f(p)$ or use generating functions.

Solution 1. The answer is $f(p) \equiv 0 \mod 3$ for $p = 3$, $f(p) \equiv 1 \mod 3$ for $p = 3k + 1$, and $f(p) \equiv -1 \mod 3$ for $p = 3k - 1$.

The case $p = 3$ is already considered, let further $p \neq 3$. For a residue i modulo p denote by $a_i(k)$ the number of Flea strategies for which she is at position i modulo p after k minutes. Then $f(p) = a_0(p-1)$. The natural recurrence is $a_i(k+1) = a_{i-1}(k)+a_i(k)+a_{i+1}(k)$, where the indices are taken modulo p. The idea is that modulo p we have $a_0(p) \equiv 3$ and $a_i(p) \equiv 0$. Indeed, for all strategies for p minutes for which not all p actions are the same, we may cyclically shift the actions, and so we partition such strategies onto groups by p strategies which result with the same i. Remaining three strategies correspond to $i = 0$. Thus, if we denote $x_i = a_i(p-1)$, we get a system of equations $x_{-1}+x_0+x_1 = 3$, $x_{i-1}+x_i+x_{i+1} = 0$ for all $i = 1, \dots, p-1$. It is not hard to solve this system (using the 3-periodicity, for example). For $p = 3k + 1$ we get $(x_0, x_1, \dots, x_{p-1}) = (1, 1, -2, 1, 1, -2, \dots, 1)$, and $(x_0, x_1, \dots, x_{p-1}) = (-1, 2, -1, -1, 2, \dots, 2)$ for $p = 3k + 2$.

Solution 2. Note that $f(p)$ is the constant term of the Laurent polynomial $(x + 1 + 1/x)^{p-1}$ (the moves to right, to left and staying are in natural correspondence with x, $1/x$ and 1.) Thus, working with power series over $\mathbb{F}_p$ we get (using the notation $[x^k]P(x)$ for the coefficient of x^k in P)

$$\begin{aligned} f(p) &= [x^{p-1}](1+x+x^2)^{p-1} = [x^{p-1}](1-x^3)^{p-1}(1-x)^{1-p} = [x^{p-1}](1-x^3)^p(1-x)^{-p}(1-x^3)^{-1}(1-x) \\ &= [x^{p-1}](1-x^{3p})(1-x^p)^{-1}(1-x^3)^{-1}(1-x) = [x^{p-1}](1-x^3)^{-1}(1-x), \end{aligned}$$

and expanding $(1 - x^3)^{-1} = \sum x^{3k}$ we get the answer.

Problem 4. Let $n > 3$ be an integer. Let Ω be the set of all triples of distinct elements of $\{1, 2, \ldots, n\}$. Let m denote the minimal number of colours which suffice to colour Ω so that whenever $1 \le a < b < c < d \le n$, the triples $\{a, b, c\}$ and $\{b, c, d\}$ have different colours. Prove that

$$\frac{1}{100} \log \log n \leqslant m \leqslant 100 \log \log n.$$

(proposed by Danila Cherkashin, St. Petersburg)

Hint: Define two graphs, one on Ω and another graph on pairs (2-element sets).

Solution. For $k = 1, 2, \ldots, n$ denote by Ω_k the set of all $\binom{n}{k}$ k-subsets of $[n]$. For each $k = 1, 2, \ldots, n-1$ define a directed graph G_k whose vertices are elements of Ω_k, and edges correspond to elements of Ω_{k+1} as follows: if $1 \leqslant a_1 < a_2 < \ldots < a_{k+1} \leqslant n$, then the edge of G_k corresponding to $(a_1, \ldots, a_{k+1})$ goes from $(a_1, \ldots, a_k)$ to $(a_2, \ldots, a_{k+1})$.

For a directed graph $G = (V, E)$ we call a subset $E_1 \subset E$ *admissible*, if E_1 does not contain a directed path $a - b - c$ of length 2. Define *b-index* $b(G)$ of the G as the minimal number of admissible sets which cover E. As usual, a subset $V_1 \subset V$ is called *independent*, if there are no edges with both endpoints in V_1; a *chromatic number* of G is defined as the minimal number of independent sets which cover V.

A straightforward but crucial observation is the following

Lemma. For all $k = 2, 3, \ldots, n$ a subset $A_k \subset \Omega_k$ is independent in G_k if and only if it is admissible as a set of edges of G_{k-1}.

Corollary. $\chi(G_k) = b(G_{k-1})$ for all $k = 2, 3, \ldots, n$.

Now the bounds for numbers $\chi(G_k)$ follow by induction using the following general

Lemma. For a directed graph $G = (V, E)$ we have

$$\log_2 \chi(G) \leqslant b(G) \leqslant 2\lceil \log_2 \chi(G) \rceil.$$

Proof. 1) Denote $b(G) = m$ and prove that $\log_2 \chi(G) \leqslant m$. For this we take a covering of E by m admissible subsets $E_1, \ldots, E_m$ and define a color $c(v)$ of a vertex $v \in V$ as the following subset of $[m]$: $c(v) := \{i \in [m] : \exists vw \in E_i\}$. Note that for any edge $vw \in E$ there exists i such that $vw \in E_i$ which yields $i \in c(v)$ and $i \notin c(w)$, therefore $c(v) \neq c(w)$. So, each color class is an independent set and we get $\chi(G) \leqslant 2^m$ as needed.

2) Denote $\chi(G) = k$ and prove that $b(G) \leqslant 2\lceil \log_2 k \rceil$. Take a proper coloring $\tau \colon V \to \{0, 1, \ldots, k-1\}$ (that means that $\tau(u) \neq \tau(v)$ for all edges $vu \in E$). For an integer $x \in \{0, 1, \ldots, k-1\}$ take a binary representation $x = \sum_{i=0}^{r-1} \varepsilon_i(x) 2^i$, $\varepsilon_i(x) \in \{0, 1\}$, where $r = \lceil \log_2 k \rceil$. Consider the following $2r$ subsets of E, two subsets $E_{i,+}$ and $E_{i,-}$ for each $i \in \{0, 1, \ldots, k-1\}$:

$$E_{i,+} = \{vu \in E \colon \varepsilon_i(\tau(v)) = 0, \varepsilon_i(\tau(u)) = 1\},$$
$$E_{i,-} = \{vu \in E \colon \varepsilon_i(\tau(v)) = 1, \varepsilon_i(\tau(u)) = 0\}.$$

Each of them is admissible, and they cover E, thus $b(G) \leqslant 2r$.

Note that $\chi(G_1) = n$, thus $b(G_1) \geqslant \log_2 n$. Actually we have $b(G_1) = \lceil \log_2 n \rceil$: indeed, if we define $\tau(v) = v - 1$ for all $v \in [n] = \Omega_1$, then the above sets $E_{i,+}$ cover all edges of G_1.

The Lemma above now yields for our number $m = \chi(G_3) = b(G_2)$ the following bounds, which are better than required:

$$b(G_2) \geqslant \log_2 \chi(G_2) = \log_2 b(G_1) = \log_2 \lceil \log_2 n \rceil$$
$$b(G_2) \leqslant 2\lceil \log_2 \chi(G_2) \rceil = 2\lceil \log_2 b(G_1) \rceil = 2\lceil \log_2 \lceil \log_2 n \rceil \rceil.$$

Remark. Actually the upper bound in the Lemma may be improved to $(1 + o(1)) \log_2 \chi(G)$ that yields $m = (1 + o(1)) \log_2 \log_2 n$.