IMC 2024

First Day, August 7, 2024

Problem 1. Determine all pairs $(a, b) \in \mathbb{C} \times \mathbb{C}$ satisfying

$$|a| = |b| = 1 \quad \text{and} \quad a + b + a\bar{b} \in \mathbb{R}.$$

(10 points)

Problem 2. For $n = 1, 2, \ldots$ let

$$S_n = \log\left(\sqrt[n^2]{1^1 \cdot 2^2 \cdot \ldots \cdot n^n}\right) - \log(\sqrt{n}),$$

where log denotes the natural logarithm. Find $\lim_{n\to\infty} S_n$.

(10 points)

Problem 3. For which positive integers n does there exist an $n \times n$ matrix A whose entries are all in $\{0, 1\}$, such that A^2 is the matrix of all ones?

(10 points)

Problem 4. Let g and h be two distinct elements of a group G, and let n be a positive integer. Consider a sequence $w = (w_1, w_2, \ldots)$ which is not eventually periodic and where each w_i is either g or h. Denote by H the subgroup of G generated by all elements of the form $w_k w_{k+1} \ldots w_{k+n-1}$ with $k \geq 1$. Prove that H does not depend on the choice of the sequence w (but may depend on n).

(10 points)

Problem 5. Let $n > d$ be positive integers. Choose n independent, uniformly distributed random points $x_1, \ldots, x_n$ in the unit ball $B \subset \mathbb{R}^d$ centered at the origin. For a point $p \in B$ denote by $f(p)$ the probability that the convex hull of $x_1, \ldots, x_n$ contains p. Prove that if $p, q \in B$ and the distance of p from the origin is smaller than the distance of q from the origin, then $f(p) \geq f(q)$.

(10 points)

After the end of contest, the solutions and preliminary results will be posted at
`https://imc-math.org.uk/?year=2024&item=problems` and
`https://imc-math.org.uk/?year=2024&item=results`.