IMC 2026

Second Day, July 30, 2026

Solutions

Problem 6.

(a) Is there a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f'(f(x)) = x$$

for every $x \in \mathbb{R}$?

(b) Is there a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f'(f(x)) = |x|$$

for every $x \in \mathbb{R}$?

(proposed by Nikolaos Kolliopoulos, University of Cyprus)

Solution.

(a) **There is no such function.** The given identity implies that the function is injective. Since it is differentiable and thus continuous, it is also strictly monotone, meaning that $f'(x)$ cannot admit both negative and positive values. The latter clearly contradicts the given identity, meaning that such a function does not exist.

(b) **We will show an example for such a function.** It is reasonable to look for a function of the form $f(x) = k|x|^a$ for some $k > 0$, which is differentiable for $a > 1$ with $f'(x) = (ka)\operatorname{sign}(x)|x|^{a-1}$ for any $x \in \mathbb{R}$. Since $f(x) \geq 0$ for each $x \in \mathbb{R}$, we see that

$$f'(f(x)) = (ka)(k|x|^a)^{a-1} = ak^a|x|^{a^2-a}$$

for all $x \in \mathbb{R}$, so all we need is to pick $a = \frac{\sqrt{5}+1}{2} > 1$ which solves the quadratic equations $a^2 - a = 1$ and thus $|x|^{a^2-a} = |x|$ for all $x \in \mathbb{R}$, and then $k = \frac{1}{a^{\frac{1}{a}}}$ so that $ak^a = 1$.

Problem 7. For a continuous function $f\colon [0,1] \to \mathbb{R}$, let $S(f)$ be the union of all straight segments in the plane joining points $(x,0)$ and $(f(x),1)$, where $x \in [0,1]$. Let $A(f)$ be the area of $S(f)$. Find the infimum of $A(f)$ over all continuous f.

(proposed by David Preiss, University of Warwick, UK)

Solution. For $s \in [0,1]$ define $f_s(x) = x + s(f(x) - x)$. Then $(f_s(x), s)$ belongs to the straight segment joining $(x,0)$ and $(f(x),1)$, so it is in $S(f)$. Since f_s is continuous, by the intermediate value theorem the straight segment joining $(f_s(0), s)$ and $(f_s(1), s)$ (which could be just a point) lies in $S(f)$. Hence the area of $S(f)$ is at least $\int_0^1 |f_s(1) - f_s(0)|\, ds = \int_0^1 |1 - cs|\, ds$ where $c = f(0) - f(1) + 1$. The latter integral is $1 - c/2$ if $c \le 1$ and $c/2 + 1/c - 1$ when $c \ge 1$. In the first case the minimum is $1/2$ (for $c = 1$) and in the second $\sqrt{2} - 1 < 1/2$ (for $c = \sqrt{2}$). So the minimum of the areas of $S(f)$ is $\sqrt{2} - 1$ which is attained, for example, when $f(x) = (1 - \sqrt{2})x$ since in this case $S(f)$ is exactly the union of the straight segments joining $(f_s(0), s)$ and $(f_s(1), s)$.

Problem 8. Let $n \geq 5$, and suppose that $A = (a_{ij})$ is a real symmetric $n \times n$ matrix such that

$$a_{ii} = 0 \quad \text{and} \quad a_{ij} \in \{-1, 1\} \text{ for } i \neq j.$$

Assume that the scalar products of any two distinct rows of A have the same value. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Prove that

$$\sum_{i=1}^{n} |\lambda_i| \geq 2n - 2$$

and determine all matrices for which equality holds.

(proposed by Slobodan Filipovski, University of Primorska, Koper)

Solution. Let c be the common scalar product of any two distinct rows, and let J be the all-ones matrix. Since every row has squared norm $n-1$ and every two distinct rows have scalar product c, we have

$$A^2 = (n - 1 - c)I + cJ. \tag{1}$$

The eigenvalues of the right-hand side are

$$(n-1)(c+1) \quad \text{and} \quad n - 1 - c \quad (\text{with multiplicity } n-1).$$

Since A is real symmetric, the matrix A^2 is positive semidefinite. Hence all eigenvalues of A^2 are nonnegative. Thus $c \geq -1$. Also, for $i \neq j$,

$$c = \sum_{k=1}^{n} a_{ik}a_{jk} = \sum_{k \neq i,j} a_{ik}a_{jk} \leq n - 2,$$

since the two omitted terms are zero and each remaining term is at most 1. Therefore

$$-1 \leq c \leq n - 2.$$

Since the eigenvalues of A^2 are $\lambda_1^2, \ldots, \lambda_n^2$, we obtain

$$\sum_{i=1}^{n} |\lambda_i| = \sqrt{(n-1)(c+1)} + (n-1)\sqrt{n-1-c}. \tag{2}$$

The right-hand side of (2) is a concave function of c on the interval $[-1, n-2]$. Its minimum is therefore attained at an endpoint. At the two endpoints its values are

$$(n-1)\sqrt{n} \quad \text{and} \quad 2n - 2,$$

respectively. Since $n \geq 5$, we have $(n-1)\sqrt{n} > 2n - 2$, and hence

$$\sum_{i=1}^{n} |\lambda_i| \geq 2n - 2.$$

Equality can occur only for $c = n - 2$. By (1),

$$A^2 = I + (n-2)J. \tag{3}$$

It follows that A commutes with J. Hence $A\mathbf{1} = r\mathbf{1}$ for some real number r, where $\mathbf{1}$ is the all-ones vector. Applying (3) to $\mathbf{1}$ gives

$$r^2\mathbf{1} = A^2\mathbf{1} = (n-1)^2\mathbf{1},$$

so $r = \pm(n-1)$. Since every row contains exactly $n-1$ entries equal to ± 1, its sum can equal $\pm(n-1)$ only if all off-diagonal entries in that row are equal. Hence every row consists entirely of 1's or entirely of -1's. By symmetry, all rows have the same sign, and therefore

$$A = J - I \quad \text{or} \quad A = I - J.$$

Conversely, these two matrices have spectra

$$\{n-1, -1, \ldots, -1\} \quad \text{and} \quad \{-(n-1), 1, \ldots, 1\},$$

respectively, and in both cases the sum of the absolute values of the eigenvalues is $2n-2$.

Problem 9. Let $a_1, a_2, \ldots$ be an infinite sequence of positive real numbers satisfying

$$a_1 + a_2 + \cdots + a_{2n-1} = a_n^2$$

for all positive integers n. Prove that $a_n \geq 2n-1$ for all positive integers n.

(proposed by Ilya I. Bogdanov, MIPT, Moscow and Aleksandr Kuznetsov, SPbU, Saint Petersburg)

Solution 1. The condition applied to $n=1$ gives $a_1 = 1$, so the desired inequality holds for $n=1$.

Denote $\delta_n = a_{n+1} - a_n$. Say that an index i is *regular* if $a_{i+1} \geq 2i+1$; otherwise i is *irregular*. Our aim is to prove that all indices are regular.

Say that our sequence is μ-*good* if $\delta_i \geq \mu$ for all irregular indices i. We will show that our sequence is μ-good for all $\mu < 2$. This yields the desired inequality; indeed, otherwise, choosing the minimum irregular index n, we get $a_{n+1} - a_n < (2n+1) - (2n-1) = 2$, so the sequence would not be μ-good for some $\mu < 2$.

Notice that $a_{n+1}^2 - a_n^2 = a_{2n} + a_{2n+1} > 0$, so $a_{n+1} > a_n$ for all n, and hence the sequence is 0-good. Now the desired statement follows from the Claim below.

Claim. If our sequence (a_n) is μ-good for some $0 \leq \mu < 2$, then it is also ν-good for $\nu = \frac{\mu}{2} + 1$.

Proof. Consider any irregular index n. Notice that $a_k - a_{n+1} \geq (k-n-1)\mu$ for every $k > n+1$; indeed, if all indices $n+1, n+2, \ldots, k-1$ are irregular, then this follows from (a_n) being μ-good. Otherwise, let i be the maximum regular index not exceeding $k-1$. Then

$$a_k - a_{n+1} = (a_k - a_{i+1}) + (a_{i+1} - a_{n+1}) \geq \mu(k-i-1) + ((2i+1)-(2n+1)) = \mu(k-i-1) + 2(i-n) > \mu(k-n-1),$$

as desired.

Therefore,

$$\delta_n(2a_{n+1} - \delta_n) = a_{n+1}^2 - a_n^2 = a_{2n} + a_{2n+1} \geq 2a_{n+1} + (2n-1)\mu,$$

This inequality easily yields $\delta_n > 1$, so in particular $a_{n+1} > 2$. Then the function $f(x) = x(2a_{n+1} - x)$ increases on $x \in [0,2]$, and in order to prove $\delta_n \geq \nu$ it suffices to show that $f(\nu) \leq 2a_{n+1} + (2n-1)\mu$. This inequality rewrites as

$$\mu a_{n+1} - \left(\frac{\mu}{2} + 1\right)^2 \leq (2n-1)\mu \iff \mu(a_{n+1} - 2n - 1) \leq \left(\frac{\mu}{2} - 1\right)^2.$$

This last inequality holds, since the left hand part is negative, while the right hand part is positive.

Solution 2. First, let us prove that the required inequality holds asymptotically, namely

Lemma. $\liminf\limits_{n\to\infty} \frac{a_n}{2n-1} \geq 1$

Proof. Let A denote this limit inferior. Clearly, the sequence is monotonically increasing. Then $a_n^2 = a_1 + a_2 + \cdots + a_{2n-1} \geq n a_n$, hence $A \geq \frac{1}{2} > 0$.

Let $\varepsilon > 0$. We know that $a_n \geq (A-\varepsilon)(2n-1)$ for all sufficiently large n. Then, from the inequality

$$a_n^2 = a_1 + a_2 + \cdots + a_{2n-1} \geq (A-\varepsilon)(1 + 3 + \ldots + 4n - 3) + O(1) = (A-\varepsilon)(2n-1)^2 + O(1)$$

we obtain $A^2 \geq A - \varepsilon$. Letting ε tend to zero, we get $A \geq 1$.

Let us make the substitution $b_n = a_n - (2n-1)$. The recurrence relation takes the form

$$b_n(b_n + 2(2n-1)) = b_1 + \ldots + b_{2n-1},$$

and the lemma implies that $\liminf\limits_{n\to\infty} \frac{b_n}{2n-1} \geq 0$. Consider $B = \inf\limits_n \frac{b_n}{2n-1}$. If $B < 0$, then the infimum is attained at some $k \in \mathbb{N}$, i.e. $b_k = B(2k-1)$ (otherwise we get a contradiction with the lemma). Then

$$B(B+2)(2k-1)^2 = b_k(b_k + 2(2k-1)) = b_1 + \ldots + b_{2k-1} > B(1 + 3 + \ldots + 4k - 3) = B(2k-1)^2.$$

This yields $B < -1$. On the other hand, $a_n > 0$ implies $B > -1$ and we get a contradiction.

Solution 2 (by Dan Carmon). Begin with observing $a_1 = a_1^2$, so $a_1 = 1$ from positivity. It follows that $a_n^2 = \sum_{i=1}^{2n-1} a_i \geq a_1 = 1$ so $a_n \geq 1 = (2n-1)^0$ for every n. Applying the same technique again gives $a_n^2 = \sum_{i=1}^{2n-1} a_i \geq 2n-1$ so $a_n \geq \sqrt{2n-1} = (2n-1)^{1/2}$. Applying this technique again and again, we will prove the following claim by induction on m:

Claim. Let $m \geq 0$, and set $c_m = 1 - 2^{-m}$. Then there is a constant $C_m > 0$ such that $a_n \geq C_m(2n-1)^{c_m}$ for all $n \geq 1$.

Moreover, we will get a recurrence relation for C_m, and show that $\lim_{m\to\infty} C_m = 1$. It would follow that $a_n \geq \lim_{m\to\infty} C_m(2n-1)^{c_m} = 2n-1$, as we are asked to show.

Proof. We have already established the $m = 0, 1$ ($c_0 = 0$, $c_1 = 1/2$) with $C_0 = C_1 = 1$. Let $m \geq 1$ and suppose $a_n \geq C_m(2n-1)^{c_m}$ for every n. Write $c = c_m$ and $C = C_m$ for brevity. Since $c < 1$, the function x^c is concave, and therefore have $x^c \geq \frac{1}{2}\int_{x-1}^{x+1} t^c dt$. Thus

$$\begin{aligned} a_n^2 &= \sum_{k=1}^{2n-1} a_k \geq C \sum_{k=1}^{2n-1} (2k-1)^c \geq \frac{C}{2} \sum_{k=1}^{2n-1} \int_{2k-2}^{2k} t^c dt = \frac{C}{2} \int_0^{4n-2} t^c dt = \frac{C}{2} \frac{(4n-2)^{c+1}}{c+1} \\ &= \frac{2^c C}{1+c}(2n-1)^{1+c}. \end{aligned}$$

Recall $c = c_m = 1 - 2^{-m}$ hence $\frac{1+c}{2} = c_{m+1}$. Taking the square root of the above inequality yields

$$a_n \geq \sqrt{\frac{2^{-2^{-m}}}{1 - 2^{-m-1}} C_m} \cdot (2n-1)^{c_{m+1}} \tag{1}$$

Which completes the inductions step for $C_{m+1} = \sqrt{B_m C_m}$, where $B_m = \frac{2^{-2^{-m}}}{1-2^{-m-1}}$.

Observe that $B_m \to 1$ as $m \to \infty$ (since $2^{-m} \to 0$), and the sequence C_m is obtained by repeatedly averaging (geometrically) the previous term with the terms of B_m. It is well known (a standard exercise in calculus 1) that this implies C_m also converges and to the same limit as B_m, as we claimed. This can be shown directly by limits calculus; another method is to apply Cesaro's theorem on geometric means to the sequence D_k defined by $D_1 = C_1$, $D_k = B_{\lceil \log_2(k) \rceil}$ (i.e., the first elements are $C_1, B_1, B_2, B_2, B_3, B_3, B_3, B_3, B_4, \ldots$), which clearly has the same limit as B_m, and each C_m is just the geometric mean of the first 2^{m-1} elements of D_k.

Problem 10. An *infinite chessboard of size* $d > 0$ is obtained by colouring the interiors of the squares of an infinite square grid of side length d alternately white and black following the usual chessboard pattern. The points belonging to the grid lines have neither colour and the grid may be translated and rotated arbitrarily in the plane.

Is it true that for any finite set of points $p_1, \dots, p_n$ in the plane, there exist $d \in (0, 1)$ and an infinite chessboard of size d such that all the points p_i lie in white squares?

(proposed by David Hruška, Czech Academy of Sciences, Prague)

Solution. We prove that the statement is true.

The standard one-dimensional Dirichlet approximation theorem says that, for every real α and positive integer Q, there are integers p, q with $1 \le q \le Q$ and $|q\alpha - p| < 1/Q$. We need its simultaneous form, sometimes called multidimensional Dirichlet approximation. Its role here is to approximate all coordinates at once using the same multiplier K:

Lemma. For real numbers $\alpha_1, \dots, \alpha_m$ and a positive integer Q, there exists an integer K with $1 \le K \le Q^m$ such that

$$|K\alpha_j - p_j| < \frac{1}{Q} \qquad (j = 1, \dots, m)$$

for suitable integers $p_1, \dots, p_m$.

Proof. Consider the $Q^m + 1$ points

$$\big(\{k\alpha_1\}, \dots, \{k\alpha_m\}\big) \in [0, 1)^m, \qquad k = 0, 1, \dots, Q^m.$$

Split $[0, 1)^m$ into Q^m half-open cubes of side length $1/Q$. By the pigeonhole principle, two of the $Q^m + 1$ points lie in the same cube; denote their indices by $k < \ell$. For $K = \ell - k$, subtracting the two coordinatewise gives integers $p_1, \dots, p_m$ satisfying the required inequalities and proves the lemma.

Let the points be $(x_1, y_1), \dots, (x_N, y_N)$ and apply the lemma with $m = 2N$ and $Q = 5$. We obtain an integer $1 \le K \le 5^{2N}$ and integers p_i, q_i such that

$$|Kx_i - p_i| < \tfrac{1}{5}, \qquad |Ky_i - q_i| < \tfrac{1}{5} \qquad (i = 1, \dots, N).$$

Thus, after multiplication by the same integer K, every coordinate is “close” to an integer.

Take an axes-aligned checkerboard of size

$$d = \frac{1}{2K} \le \frac{1}{2} < 1,$$

translated by $d/2$ in each coordinate, i.e. with the origin being the center of one square and let us colour this square white. The grid lines then satisfy $x/d + 1/2 \in \mathbb{Z}$ and $y/d + 1/2 \in \mathbb{Z}$ and a point (x, y) which does not belong to any grid line is white precisely when

$$\left\lfloor \frac{x}{d} + \frac{1}{2} \right\rfloor + \left\lfloor \frac{y}{d} + \frac{1}{2} \right\rfloor \equiv 0 \pmod 2.$$

Write $Kx_i = p_i + \delta_i$ with $|\delta_i| < 1/5$. The choice $d = 1/(2K)$ makes the main term $2p_i$ even, while the translation by $d/2$ keeps the small error inside the same square. Indeed,

$$\frac{x_i}{d} + \frac{1}{2} = 2Kx_i + \frac{1}{2} = 2p_i + \underbrace{2\delta_i + \frac{1}{2}}_{=:\varepsilon_i}.$$

Since $|2\delta_i| < 2/5$, we have

$$0 < \varepsilon_i < 1,$$

so

$$\left\lfloor \frac{x_i}{d} + \frac{1}{2} \right\rfloor = 2p_i,$$

which is even. The same argument gives

$$\left\lfloor \frac{y_i}{d} + \frac{1}{2} \right\rfloor = 2q_i,$$

also even. Hence the sum of the two square indices is even, so every point lies in a white square. The strict inequalities $0 < \varepsilon_i < 1$ also show that no point lies on a grid line.