Problem 1 – Marking Scheme

Solution 1: Official Solution

- Stating one of the equalities $\cos(x) = \sin(\frac{\pi}{2} - x)$ or $\sin(x) = \cos(\frac{\pi}{2} - x)$ and arriving to $\sin(\sin(x)) = \sin(\frac{\pi}{2} - \cos(x))$ or $\cos(\cos(x)) = \cos(\frac{\pi}{2} - \sin(x))$ **(2 pt)**
- Stating $\sin(x) = \sin(y) \iff y = x + 2\pi k$ or $y = \pi - x + 2\pi k$ (or $\cos(x) = \cos(y) \iff y = x + 2\pi k$ or $y = -x + 2\pi k$) and arriving to $\sin(x) \pm \cos(x) = \pm\frac{\pi}{2} + 2\pi k$ **(1+1 pt)**
  - Only 1 pt is awarded if not both cases are stated.
  - Only 1 pt is awarded if the student forgets $+2\pi k$ (the 2π periodicity).
- Arguing that a solution may only exist when $k = 0$ and arriving to $|\sin(x) \pm \cos(x)| = \frac{\pi}{2}$ **(2 pt)**
  - No points are awarded if in the previous item the student forgets the $+2\pi k$ term.
- Proving or stating as common knowledge that $|\sin(x) \pm \cos(x)| \leq \sqrt{2}$ **(1 pt)**
  - This point is also awarded for proving another upper bound which will later lead to a contradiction.
- Using the previous upper bound estimate to conclude that $|\sin(x) \pm \cos(x)| = \frac{\pi}{2}$ has no real solutions **(3 pt)**
  - Only 1 point is awarded if only one case is treated.
  - Saying that the second case is similar to the first one awards all 3 pts.

Solution 2: Factorisation

- Writing $\sin(\sin(x)) - \cos(\cos(x)) = 2\cos\left(\frac{\pi}{4} + \frac{\sin(x)-\cos(x)}{2}\right) \cdot \sin\left(-\frac{\pi}{4} + \frac{\sin(x)+\cos(x)}{2}\right)$ **(2 pt)**
- Stating that this difference is 0 iff $\cos(\dots) = 0$ or $\sin(\dots) = 0$, which implies: $\sin(x) \pm \cos(x) = \frac{\pi}{2} + 2\pi k$ **(1+1 pt)**
- Arguing that a solution may only exist when $k = 0$ and arriving to $|\sin(x) \pm \cos(x)| = \frac{\pi}{2}$ **(2 pt)**
- Proving or stating as common knowledge that $|\sin(x) \pm \cos(x)| \leq \sqrt{2}$ **(1 pt)**
- Using the previous upper bound to conclude $\sin(x) \pm \cos(x) = \frac{\pi}{2}$ has no real solutions. **(3 pt)**

Solution 3: Taylor estimates

- Stating the estimates $\sin(x) < x$ for $x \in (0, \infty)$ and $\cos(x) > 1 - \frac{x^2}{2}$ for $x \in \mathbb{R} \setminus \{0\}$ **(1+1 pt)**
- Writing $y = \sin(x)$ arriving to $\cos(\pm\sqrt{1 - y^2}) = \sin(y)$ **(2 pt)**
- For $y \in (0, 1) \cup (1, \infty)$ applying the Taylor estimates and obtaining the contradiction **(3 pt)**
- Dealing with the cases $y = 1$ and $y \leq 0$ **(3 pt)**

IMC 2026 Problem 2 marking scheme

Solution 1

Step 1 (2 points):

$$0 = \operatorname{tr}(A^T A + BB^T - AB - BA)$$

Step 2 (2 points):

$$= \sum_{i,j} a_{ij}^2 + \sum_{i,j} b_{ji}^2 - 2\sum_{i,j} a_{ij} b_{ji}$$

Step 3 (2 points):

$$= \sum_{i,j} (a_{ij} - b_{ji})^2$$

Step 4 (2 points):

$$\implies B = A^t$$

Step 5 (2 points):

$$\implies AB = BA$$

Solution 2

Step 1 (1 point): Introduce $X = A - B^T$ and $X^T X$ or some equivalent.

Step 2 (2 points):

$$X^T X = AB - A^T B^T$$

Step 3 (1 point): Conclude that $\lambda_i(XX^T) \geq 0$.

Step 4 (2 points): Since $\operatorname{tr}(AB - A^T B^T) = 0$, conclude that $\lambda_i(XX^T) = 0$.

Step 5 (2 points):

$$\implies B = A^t$$

Step 6 (2 points):

$$\implies AB = BA$$

Alternative version of steps 3–5: instead of looking at eigenvalues, recognize that $\|X\|^2 = \operatorname{tr}(AB - A^T B^T) = 0$, where $\|\cdot\|$ is the Euclidean matrix norm .

Remarks

Not worth any points: showing / noticing / stating that $AB - BA$ is symmetric and solution under the assumption that A and B are symmetric themselves.

A solution under the assumption that A and B are normal may be up to 5 points: 1 point for introducing $X = A - B^T$ and computing $X^T X$ (which is automatically 0), then 2 more points for proving that $XX^T = 0$ implies $X = 0$ and then 2 more points for substituting $B = A^T$ back to the original equation.

IMC 2026 – Problem 3 Marking Scheme

Total: 10 points

Let $A \subseteq [n]$ be the set of cards placed in the increasing stack, and let $\pi(A)$ denote the resulting final order. An *overlap* is a pair of distinct choices of A that produce the same final order. Let E_n denote the number of overlaps. Write $F_0 = 0$, $F_1 = 1$, and $F_{m+1} = F_m + F_{m-1}$.

(i) Initial count and reduction to overlaps. **[1 pt]**

Noting that there are $2^n - 2$ valid two-colourings, since neither stack may be empty, and that the required number is

$$2^n - 2 - E_n.$$

(ii) Structure and multiplicity of overlaps. **[3 pt]**

Proving that if $\pi(A) = \pi(A')$ with $A \neq A'$, then the two splits differ by moving exactly one critical card from one stack to the other.
Deducing that no final order is produced by more than two splits, so every overlap is subtracted exactly once.

(iii) Counting the overlaps. **[4 pt]**

Identifying, in each of the two possible cases, the forced part of an overlapping pair up to the critical card, and expressing the number of overlapping pairs using binomial coefficients:

$$E_n = \sum_{r=1}^{\lfloor (n-1)/2 \rfloor} \binom{n-r-1}{r} + \sum_{s=2}^{\lfloor n/2 \rfloor} \binom{n-s-1}{s-1}.$$

Any equivalent correctly indexed binomial expression receives full credit.

(iv) Summing the overlap count. **[1 pt]**

Using the standard Fibonacci binomial identity to obtain

$$\sum_{r=1}^{\lfloor (n-1)/2 \rfloor} \binom{n-r-1}{r} = F_n - 1, \qquad \sum_{s=2}^{\lfloor n/2 \rfloor} \binom{n-s-1}{s-1} = F_{n-1} - 1,$$

and hence

$$E_n = F_{n+1} - 2.$$

(v) Correct final answer. **[1 pt]**

Concluding that the number of distinct final orders is

$$(2^n - 2) - (F_{n+1} - 2) = \boxed{2^n - F_{n+1}}.$$

Notes.

- A solution consisting only of checks for specific values of n earns +0 pt.
- Forgetting the non-emptiness condition incurs no additional deduction: −0 pt.
  (Any consequent error is already reflected in the relevant criterion above.)
- Minor computational flaws: -1 pt.
- Observing that the number of solutions starting with "1" equals the number of solutions of length $n - 1$ with the order reversed: +1 pt.

Problem 4 – Marking Scheme

- $x_n \downarrow -1$ **(1 p)**
- $\lim_{n\to\infty} x_n = 0$ **(1 p)**
- Evaluation of the limit $\lim_{n\to\infty} \frac{x_n^2}{\ln n}$ **(3 p)**
- Application of the Stolz theorem for convergence **(3 p)**
- Use of the Taylor expansion of arctan **(2 p)**

Problem 5 – Marking Scheme

NO POINTS FOR THE FOLLOWING:

- Bound that can be obtained by interpolating on $\{-A, \ldots, -1, 1, \ldots, B\}$

  e.g.

  - $\deg p \leq A + B$
  - only proves for $\frac{A}{B} \in [C, D]$ (for fixed C, $D \in \mathbb{R}_+$)

- Mentioning Chebyshev polinomials, Fejér kernel (without using them in a way relevant to the solution).

PARTIAL / WEAKER RESULTS:

- $\deg p \leq C(A + B)$ for $C < 1$ — ≥ 1 point
- Essential ideas for a full solution. — ? points