

Marking scheme for IMC 2026, Day 2, Problem 6

Part a is worth 5 points and Part b is worth 5 points.

Part a:

- (Z1) f is differentiable $\Rightarrow f$ is continuous 0 **points**
- (A1) f is injective 1 **point**
- (A2) continuous + injective $\Rightarrow$ monotone 2 **points**
- (A3) f' has a single sign 1 **point**
- (A4) deduction from $f'(f(x))$ being both > 0 and < 0 1 **point**
- (badA1A2) instead of (A1)+(A2), only says injective $\Rightarrow$ monotone 2 **points** instead of 3
- (X) A solution for part (a) that assumes f is twice differentiable 1 **point**
- A1, A2, A3, A4 are additive.
X is not additive with anything.

Part b:

Version 1

- (B1) Consider $f(x) = c|x|^a$ 2 **points**
- weaker versions:
- Consider only $f(x) = cx^a$ 1 **point**
 - Consider only $f(x) = |x|^a$ 1 **point**
- (B2) Computation of c, a 2 **points**
- (B3) Finalization (differentiable at 0, no sign errors) 1 **point**

Version 2

- (C1) guess the correct function 4 **points**
- (C2) Verification: equation hold for every $x \in \mathbb{R}$ and differentiability 1 **point**

Comments:

- (Y1) does (B1) + (B2) but writes it to part (a) instead of part (b) 3 **points** instead of 4
- (Y2) does (weaker B1) + (B2) but writes it to part (a) instead of part (b) 3 **points**

Alternative solution for PART B:

An odd function, if $x \geq 0$ then $f(x) = cx^a$ with the same a and c as in the official solution.
For $x < 0$, $f(x) = -f(-x)$.

Marking scheme for problem 7

(G) Geometric Scheme

Denote $A = (0, 0)$, $B = (1, 0)$, $C = (f(0), 1)$, $D = (f(1), 1)$. All items below are additive with each other, except items (G4), (G4.1), (G4.2), of which only one is awarded.

(G1) *Lemma*: $S(f)$ contains either (T) convex trapezium $ABDC$ if $f(0) \leq f(1)$ or (H) concave hourglass $ABDC$ if $f(0) \geq f(1)$ **4 points**

The full 4 points are awarded only if the lemma is fully stated and fully proven. See possible deductions and explanations below.

(G2) Giving a lower bound on the area in case (T) **1 point**
e.g. bounding it by $\frac{1}{2}$, or by the area of a triangle, or by the area of an hourglass.

(G3) Proving the area of an hourglass is at least $\sqrt{2} - 1$ **3 points**
• Finding a *critical* hourglass by differentiation, without verifying it is *minimal* **-1 point**

(G4) Giving a correct example and *proving* it attains the lower bound **2 points**
(G4.1) Providing a correct example without a complete proof **1 point**
(G4.2) *Claiming* that for linear functions, the shapes from the Lemma are *exactly* $S(f)$ **1 point**

(G') Geometric' Scheme

Denote $C' = (\min f, 1)$, $D' = (\max f, 1)$. The scheme is the same as the Geometric Scheme, except items (1)+(2) are replaced with

(G'1) *Lemma'* $S(f)$ contains (H') the concave hourglass $ABC'D'$ **5 points**

(D) Partial credit / deductions from the Lemmas

(D1) Treating only the (T) case (e.g. by assuming $f(0) < f(1)$, implicitly or incorrectly) $\times \frac{1}{2}$ **points**

(D2.1) Stating Lemma, but without any proof $\times \frac{1}{2}$ **points**

(D2.2) A flawed proof of Lemma (e.g. claiming that it follows from IVT/continuity applied to the wrong function) **-1 point**

(D3) Claiming Lemma with $S(f)$ being *exactly* the shape, instead of $\supseteq$ $\times 0$ **points**

(D'2.1) Stating Lemma', but without any proof **-3 points**

(D'2.2) Stating Lemma', with partial proof **-2 or -1 points**

The relevant deductions are applied to the score in order. They are cumulative, except for subitems of (D2) and (D'2) with each other. Thus any appearance of (D3) results in 0 points. Otherwise, (D1) would result in 2 points, (D2.1) in 2, (D2.2) in 3, and (D1) together with either subitem of (D2) in 1.

(I) Integral Scheme

All items below are additive. Items (I1.X) together correspond to item (G1) above. Other (In) correspond to (Gn), together with any possible deductions or partial credit.

- (I1.1) Considering the level sets of $S(f)$, and claiming they contain the intervals $[f_t(0), f_t(1)]$ 1 point
- (I1.2) Proving the above claim 2 points
- (I1.3) Writing down $A(f) \geq \int_0^1 |\ell(t)|$, $\ell(t) = 1 - t(1 + f(0) - f(1))$ 1 point
- (I2) Considering the case $1 + f(0) - f(1) \leq 1$, showing $\ell(t) \geq 0$ and bounding $A(f) \geq \frac{1}{2}$. 1 point
- (I3) Considering the case $1 + f(0) - f(1) \geq 1$, correctly splitting the integral by the sign of $\ell(t)$ and bounding $A(f) \geq \sqrt{2} - 1$ 3 points
- (I4) Giving a correct example and *proving* it attains the lower bound 2 points

(N) Nonadditive Scheme

1 nonadditive point is available for an *inspiring picture*, if no other point is available in any other scheme. An *inspiring picture* is defined as one of the following:

- A diagram of an hourglass, where the corners are labeled $0, 1, f(1), f(0)$ in order.
- A diagram of an hourglass, where the corners are labeled $0, 1, \min f, \max f$ in order.
- Unlabeled diagrams of *both* an hourglass and a a trapezium, appearing on the same page.

(Z) Zero points

Giving an example of a function where $A(f) = \frac{1}{2}$ (e.g. a constant function) 0 points

Problem 8 grading scheme

- Computing A^2 — 2 pts (no diagonal — 1 pt)
- Finding eigenvalues of A^2 — 1 pt
- $\sum |\lambda_i| = \sqrt{(n-1)(c+1)} + (n-1)\sqrt{n-1-c}$ — 1 pt
- Minimizing it — 3 pts

Equality:

- Providing 1 example without checking — 0 pts
 and checking — 1 pt
- Proving that every row is 1 or -1 — 2 pts
 (but if not shown same sign for all rows — 1 pt)
- Missing one of the examples in full solution:

deduct
1 pt

Solution 1:

$$a_1 = 1 \quad 0 \text{ pt}$$

$$a_{2n} + a_{2n+1} = a_{n+1}^2 - a_n^2 \quad 0 \text{ pt}$$

partial cases
 $n=1, n=2$ or $n=3$ 0 pt

$$a_n \uparrow \quad 1 \text{ pt}$$

$$a_n \geq n \quad 3 \text{ pt (not summarized with previous point)}$$

$$\inf \frac{a_n}{2n-1} \neq \frac{1}{2} \quad 3 \text{ pt}$$

Solving equation for
 infinity and proving 4 pt

$$a_n \geq 2n-1 \quad 6 \text{ pt}$$

$$a_n \geq 2n-2$$

$$a_n \geq \sqrt{2n-1}$$

0 pt.

$$a_n \geq C_n(2n-1), \text{ where}$$

$$C_n \rightarrow 1$$

(-3 pt)

Solving the equation:

$$a_n^2 - a_n - (a_1 + \dots + a_{n-1} + a_{n+1} + \dots + a_{2n-1}) = 0$$

0 pt

Solution (with integrals):

$$a_n \geq \sqrt{2n-1}$$

0 pt

$$a_n \geq \sqrt{2n-1} +$$

attempts to estimate
 with integrals and sums

2 pt.

(pg)

Problem 10: Marking Scheme

Total: 10 marks

Problem. Given finitely many points $p_1, \dots, p_N \in \mathbb{R}^2$, prove that there exist $d \in (0, 1)$ and a translated and rotated infinite chessboard of square side length d such that every p_i lies in the interior of a white square.

The official solution uses a simultaneous form of Dirichlet's approximation theorem. Rotation is not needed: an axis-parallel board suffices.

Marking Scheme

A fully correct solution is awarded 10 marks.

For minor errors we deduct 1 mark.

Partial progress is marked as follows:

- **(A0) 0 marks:** Merely mentioning Kronecker's theorem, Dirichlet's theorem, or "simultaneous approximation", without a correct statement and application, earns no credit.
- **(A1) 2 marks:** A complete solution only for the special case $p_i \in \mathbb{Q}^2$ for every i .
- **(A2) 2 marks:** A complete solution only for the special case in which the relevant coordinates are $\mathbb{Q}$ -linearly independent, using Kronecker's theorem.
- **A1 and A2 are NOT ADDITIVE.** Award at most 2 marks in total if a script gives both special cases but does not cover arbitrary rational dependencies.
- **0 marks on its own:** Merely considering $\lfloor x/d \rfloor \pmod{2}$ without using it in a valid construction.
- **0 marks on its own:** Merely reducing the problem to one dimension, without obtaining a common scale that works simultaneously for both coordinates of every point.